

POSTERIOR CONVERGENCE FOR APPROXIMATED UNKNOWN IN NON-GAUSSIAN STATISTICAL INVERSE PROBLEMS

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ABSTRACT. The statistical inverse problem of estimating the probability distribution of an infinite-dimensional unknown given its noisy indirect observation is studied in the Bayesian framework. In practice, one often considers only finite-dimensional unknowns and investigates numerically their probabilities. As many unknowns are function-valued, it is of interest to know whether the estimated probabilities converge when the finite-dimensional approximations of the unknown are refined. In this work, the generalized Bayes formula is shown to be a powerful tool in the convergence studies. With the help of the generalized Bayes formula, the question of convergence of the posterior distributions is returned to the convergence of the finite-dimensional (or any other) approximations of the unknown. The approach allows many prior distributions while the restrictions are mainly for the noise model and the direct theory. Three modes of convergence of posterior distributions are considered – weak convergence, setwise convergence and convergence in variation. The convergence of conditional mean estimates is studied. Several examples of applicable infinite-dimensional non-Gaussian noise models are provided, including a generalization of the Cameron-Martin formula for certain non-Gaussian measures. Also, the well-posedness of Bayesian statistical inverse problems is studied.

1. INTRODUCTION

Statistically oriented infinite-dimensional inverse problems are often described as problems where one wants to estimate an unknown function given its randomly perturbed indirect observation [18, 40, 49, 97, 112, 141]. We prefer the following description which suits well in the Bayesian framework.

The statistical inverse problem is to estimate the probability distribution of the unknown given its randomly perturbed indirect observation.

In this paper, the unknown X and its observation Y are modeled as random mappings from a complete probability space (Ω, Σ, P) into some locally convex Souslin topological vector spaces F and G equipped with their Borel σ -algebras \mathcal{F} and \mathcal{G} , respectively. Recall, that a Souslin space is a Hausdorff topological space that is an image of a complete separable metric space under a continuous mapping. The observations are taken to be of the form $Y = L(X) + \varepsilon$, where ε represents random noise, ε and X are statistically independent, and $L : F \rightarrow G$ is a continuous mapping. The image measure $\mu_X := P \circ X^{-1}$ on F is called the prior distribution, and it represents our beliefs about the unknown without any given observations.

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Typically, we are given a sample $Y(\omega_0) = L(X(\omega_0)) + \varepsilon(\omega_0)$, which is produced by an unknown $X(\omega_0)$ and a perturbation $\varepsilon(\omega_0)$ for some $\omega_0 \in \Omega$. Ultimately, we pursue after the probability measure $U \mapsto 1_U(X(\omega_0))$ defined on the Borel sets $U \subseteq F$. This measure would determine the unknown $X(\omega_0)$ uniquely since F is a Hausdorff space, which implies that the singletons are closed sets and belong therefore to the Borel σ -algebra \mathcal{F} . We get a simple approximation of the function $\omega \mapsto 1_U(X(\omega))$ on the basis of the given $Y(\omega_0)$ by taking its orthogonal projection from $L^2(\Omega, \Sigma, P)$ onto $L^2(\Omega, \sigma(Y), P)$, where $\sigma(Y) = Y^{-1}(\mathcal{G})$ denotes the σ -algebra generated by Y . Recall, that for any $f \in L^2(\Omega, \Sigma, P)$, this projection coincides P -almost surely with the conditional expectation $\mathbf{E}[f|\sigma(Y)]$ of f given the σ -algebra generated by Y (see [39]). Moreover, there exists a measurable real-valued function λ_f on G such that $\lambda_f(Y(\omega)) = \mathbf{E}[f|\sigma(Y)](\omega)$ P -almost surely. We take $\mathbf{E}[1_U(X)|\sigma(Y)](\omega_0)$ (or more precisely, $\lambda_{1_U}(Y(\omega_0))$) as an estimate of the probability that the unknown $X(\omega_0)$ belongs to the set $U \in \mathcal{F}$.

When the mappings $U \mapsto \mathbf{E}[1_U(X)|\sigma(Y)](\omega_0)$ form a probability measure on (F, \mathcal{F}) , which is denoted here with $\mu(U, Y(\omega_0))$, this measure is called the posterior distribution of X given a sample $Y(\omega_0)$ of Y . From the posterior distribution one may extract information about the unknown $X(\omega_0)$. For example, the posterior mean may serve as an estimate of the unknown.

The above estimation of the probability distribution of the unknown is generally known as the statistical inverse theory (also known as the statistical inversion or the Bayesian inversion). We postpone a literature review on the statistical inverse theory to Section 1.4. The present paper concentrates on the following three topics in the statistical inverse theory inspired by a paper of Lassas et al [96].

- (i) Applicability of the generalized Bayes formula for statistical inverse problems in locally convex Souslin topological vector spaces.
- (ii) Well-posedness of the Bayesian statistical inverse problem.
- (iii) Convergence of posterior distributions and posterior means for approximated unknowns. Especially, finding conditions that guarantee the convergence of the posterior distributions when the corresponding approximated prior distributions converge.

1.1. Case (i): The generalized Bayes formula. When X and Y have continuous probability densities with respect to the Lebesgue measure, the conditional expectations lead to the Bayes formula

$$(1) \quad D(x|y)D(y) = D(x, y) = D(y|x)D(x)$$

which defines the unique continuous posterior probability density $D(x|y)$ for any occurred observation y such that $0 < D_Y(y) < \infty$ (see [75]). In (1), the functions $D(x)$, $D(y)$, and $D(x, y)$ denote the probability densities of $P \circ X^{-1}$, $P \circ Y^{-1}$, and $P \circ (X, Y)^{-1}$ at x , y , and (x, y) , respectively. If the observation is of the form $Y = L(X) + \varepsilon$, where X and the noise ε are statistically independent, then the conditional density of Y given $X = x$ has the special form $D(y|x) = D_\varepsilon(y - L(x))$, where D_ε is the continuous probability density of the noise ε .

The availability of the conditional density $D(y|x)$ from the relationship between unknowns and observations is the key element for the statistical inverse theory. It makes the expression of the posterior density $D(x|y)$ explicit, opening the way for exploring the posterior distribution numerically. Unfortunately, infinite-dimensional

probability measures lack probability density functions since there is no infinite-dimensional Lebesgue measure. Instead of (1), we have

$$(2) \quad \mathbf{E}[\mathbf{E}[1_U(X)|\sigma(Y)]1_V(Y)] = P(X \in U \cap Y \in V) = \mathbf{E}[1_U(X)\mathbf{E}[1_V(Y)|\sigma(X)]]$$

for all Borel sets $U \in \mathcal{F}$ and $V \in \mathcal{G}$. The distributions of X , Y , and (X, Y) are, in principle, known. However, determining $\mathbf{E}[1_U(X)|\sigma(Y)]$ explicitly from the first equality in (2) is in general a hard task, where an explicit expression of the distribution of (X, Y) is helpful, as in the case of linear Gaussian problems [100, 102, 105]. On the other hand, the second equality in (2) looks easy enough. For instance, the dominated convergence of simple functions to the exponential function shows that

$$\mathbf{E}[e^{i\langle X, \phi \rangle + i\langle Y, \psi \rangle}] = \mathbf{E}[e^{i\langle X, \phi \rangle} \mathbf{E}[e^{i\langle Y, \psi \rangle} | \sigma(X)]]$$

for all ϕ and ψ in the dual spaces F' and G' , respectively. This suggests that, after verifying some measurability conditions, we may take

$$\mathbf{E}[1_V(Y)|\sigma(X)](\omega) = \mu_{\varepsilon+L(X(\omega))}(V)$$

P -almost surely since X and ε are statistically independent. Does knowing the conditional probabilities $\mathbf{E}[1_V(Y)|\sigma(X)](\omega)$ help in determining the posterior distribution? The answer is positive in some cases. If the σ -algebra \mathcal{G} in question is countably generated and the conditional distributions of Y given X are regular and P -almost surely absolutely continuous with respect to some fixed σ -finite measure λ on G (i.e. they are dominated by λ), then the generalized Bayes formula

$$(3) \quad \mu(U, y) = \frac{\int_U \frac{d\mu_{Y|X}(\cdot, x)}{d\lambda}(y) d\mu_X(x)}{\int_F \frac{d\mu_{Y|X}(\cdot, x)}{d\lambda}(y) d\mu_X(x)}$$

is known to hold for $U \in \mathcal{F}$ and μ_Y -almost every given observation $Y = y$ such that the denominator is finite and non-zero [79, 128]. In (3), it is required that the Radon-Nikodym densities $\frac{d\mu_{Y|X}(\cdot, x)}{d\lambda}(y)$ of the conditional measure $\mu_{Y|X}(\cdot, x)$ of Y given $X = x$ with respect to λ are jointly measurable. This is sometimes achieved by defining the Radon-Nikodym densities with the help of a fixed joint density as is done in [128]. In (3), the form of Y is allowed to be more general than in our restricted case of $Y = L(X) + \varepsilon$, where the posterior distribution has, for suitable L, X , and ε , the form

$$(4) \quad \mu(U, y) = \frac{\int_U \frac{d\mu_{Y+L(X)}}{d\lambda}(y) d\mu_X(x)}{\int_F \frac{d\mu_{Y+L(X)}}{d\lambda}(y) d\mu_X(x)}$$

for all $U \in \mathcal{F}$ and μ_Y -a.e. $y \in G$ such that the denominator is finite and non-zero. When the Radon-Nikodym densities in (4) are known, the posterior distribution on F has an explicit representation for all admissible $y \in G$.

In statistical inverse problems, the generalized Bayes formula for function-valued unknowns has been used before in the case of finite-dimensional noise models that have probability density functions with respect to the Lebesgue measure [25, 44, 97, 137] and in the case of infinite-dimensional Gaussian noise models, using in (4) the Cameron-Martin formula [63, 96, 137]. The starting point in [25, 26, 137] is that the posterior distribution is assumed to have Radon-Nikodym density with respect to the prior distribution. Therefore, a similar formula like (3) is used in [25, 26, 137], but not derived. In [63, 96, 97], the unknown and the noise are statistically independent. The same seems to be the case in some examples in [25, 26, 137] but the fact is not emphasized.

Fitzpatrick [44] studied (separable) Banach-space valued unknowns, and wrote the expression (4) in the case of finite-dimensional observations. As a concrete example, he used a Gaussian prior distribution on $C([0, 1])$ in the ill-posed inverse problem of determining the function q in the differential equation $-(qu')' = f$ on $(0, 1)$ from finitely many noisy values of the solution u satisfying the Dirichlet boundary condition. Lassas and Siltanen [97] used the generalized Bayes formula for certain prior random variables on $C([0, 1])$ and assumed the finite-dimensional noise to be Gaussian. Lassas et al [96] and Helin [63] had emphasis on edge-preserving prior distributions and used linear forward theory with Gaussian noise, but they allowed in (4) also other separable Banach and Hilbert space-valued unknowns, respectively. The forward mapping L was assumed to be linear in [63, 96, 97]. Cotter et al [25] studied the case of finite-dimensional observations and Banach space-valued unknowns, and required L to be measurable. Stuart [137] assumed L to be locally Lipschitz continuous and assumed finite-dimensional or Gaussian noise. Stuart allowed prior distributions that are absolutely continuous with respect to some Gaussian measure. Theorem 4.1 in [137] is an abstract generalization towards allowing certain infinite-dimensional non-Gaussian noise distributions but the identification of the used notation to any statistical inverse problem is omitted. The same approach is used in [26].

In the present paper, we provide (abstract) assumptions on the forward theory and the noise that are sufficient for the generalized Bayes formula in the case of statistical inverse problems in locally convex Souslin topological vector spaces. However, such a generalization is not particularly novel by itself, and the generalized Bayes formula is treated in this work as an important tool for achieving other results. For example, the study of Case (iii) exploits the generalized Bayes formula.

In this work, we allow infinite-dimensional noise models (similarly as in [63, 94, 96, 116]). One may ask, what are the benefits of such models because any feasible measuring instrument produces only finite-dimensional observations. For example, an analog-to-digital converter performs the weighted averaging and quantization of the signal; an X-ray imaging device has a finite number of projection angles and a limited resolution of the projection images. In this light, there is no immediate need for infinite-dimensional noise models. However, changes in the measuring instrument can lead to different posterior distributions and one may wish to choose the best finite-dimensional measurement configuration for the problem. As noted in [96], the mathematical formulation of the infinite-dimensional noise model, when possible, may be helpful, as it provides an overall framework for the studies. For some noise sources there even exists physically motivated infinite-dimensional noise models, like the model of the thermal noise in electric circuits, which arises from the thermal motion of the charge carriers.

Particular emphasis in this work is on finding tools for dealing with non-Gaussian noise in infinite-dimensional statistical inverse problems. There are three reasons why the Gaussian noise model is not satisfactory.

- (1) Noise does not always follow well enough a Gaussian distribution. In Section 5.4 we discuss the appearance of α -stable noise in statistical inverse problems. An evaluation of finite-dimensional noise models in medical imaging can be found in [57].
- (2) Model approximations – which were studied first by Kaipio and Somersalo [76] for finite-dimensional observations – can also produce non-Gaussian errors (cf. Remark 14).

- (3) Some noise statistics may not be exactly known. In statistical inverse theory the inaccuracies in the noise model are further modeled with hierarchical distributions (see Section 5.5 for a special case).

A wrong noise distribution may cause poor performance of the estimators of the unknown.

In Section 5, we are able to derive (with the help of the generalized Bayes formula) explicit posterior distributions in some new cases where the noise has non-Gaussian infinite-dimensional distribution (Sections 5.3–5.7). It turns out that in some cases (Section 5.4) the posterior distribution has a simple expression for the infinite-dimensional observations but not for the truncated finite-dimensional observations. As a further motivation for the study of infinite-dimensional noise models, we suggest that the solutions of infinite-dimensional Bayesian problems may give rise to new numerically feasible, but non-Bayesian, approximations of the finite-dimensional posterior distributions.

1.2. Case (ii): Well-posedness of the Bayesian statistical inverse problem.

The projection operator from $L^2(\Omega, \Sigma, P)$ onto $L^2(\Omega, \sigma(Y), P)$ determines posterior probabilities $\mu(U, Y(\omega))$ only up to P -almost every $\omega \in \Omega$. The uniqueness of the posterior distribution for a given $y \in \mathcal{R}(Y)$ is therefore unsettled (note that such form of nonuniqueness has nothing to do with the uniqueness of the deterministic inverse problem of recovering x_0 from $L(x_0)$). The nonuniqueness is fairly well understood in Gaussian linear problems [94, 100, 102, 105, 133], where the posterior mean is known to be determined up to a set of probability zero, but has received limited attention in the general case. For a Bayesian scientist, such nonuniqueness is disconcerting. Two Bayesians using the same prior distribution and the same observations can, in principle, have different posterior distributions for some observations (in a set of probability zero). One aim of the present work is to make the two Bayesians agree on the form of their posteriors for a given $y \in G$, at least in some special cases. In Theorem 2.4, we first carefully identify the nonuniqueness of the posterior distributions in locally convex Souslin topological vector spaces by adopting a new concept, the essential uniqueness, from the theory of conditional measures to the statistical inverse theory. Then, we apply a choice first made in [96] and appearing also in [25, 26, 63, 137], which is to work, if possible, with a fixed version of the posterior distribution depending continuously on observations in certain sense. Evans and Stark suggested even earlier that certain non-uniqueness problems with conditional expectations could be avoided by using dominated probabilities (see Remark 3.7 in [40]).

The original part of this work begins in Section 2.2, where we achieve partial uniqueness of those posterior distributions that depend continuously on observations in the sense that posterior probabilities of Borel sets depend continuously on observations (cf. Theorem 2.7). The partial uniqueness gives an unambiguous meaning to the posterior distribution at a fixed observation. Moreover, it shows that then the Bayesian statistical inverse problem is well-posed – there exists a unique posterior distribution that depends continuously on the observations. The method of using continuous probability densities is widely used in the finite-dimensional case (see [75]), but seems not to have been taken before within the abstract infinite-dimensional problems.

The posterior distributions are further studied in Theorem 2.8, where it is shown that the continuous dependence of the posterior probabilities of Borel sets on the

observations implies the absolute continuity of the posterior distribution with respect to the prior distribution. We remark that Theorem 2.8 clarifies some of the differences between the undominated and the μ_X -a.s. dominated cases. Indeed, if F and G are Polish vector spaces, a result of Macci [103] says that the absolute continuity of μ_Y -almost all posterior distributions with respect to the prior distribution is equivalent to the absolute continuity of the measures $\mu_{\varepsilon+L(x)}$ with respect to the measure μ_Y for μ_X -a.e. $x \in F$. Hence, the continuous dependence of the posterior probabilities of Borel sets on observations is possible only when the measures $\mu_{\varepsilon+L(x)}$ are dominated by some σ -finite measure for μ_X -a.e. $x \in F$. What does this mean for the undominated cases? The posterior probability of at least one Borel set will be discontinuous as a function of observations. Hence, the corresponding Bayesian problem is ill-posed – small perturbations of the given sample can lead to large perturbations of some posterior probabilities. The ill-posedness in the linear Gaussian statistical inverse problems has been considered before by Florens and Simoni [46, 133], who noted that the posterior mean in the Gaussian linear case can be ill-posed. Florens and Simoni also showed that the regularizing effect of the prior distribution has a limited power in such a case. They suggested using an additional Tikhonov regularization in the Gaussian linear case in order to obtain approximations of the posterior means that depend continuously on the observations.

We note that the worst-case scenario for discontinuous posterior distributions on complete separable metric spaces is somewhat characterized in [16], where it is proved that either the set of all $y \in G$ such that the posterior distributions $\mu(\cdot, y)$ are mutually singular is (at most) countable or there exists a non-empty compact perfect set $C \in G$ and a Borel set $B \in F \times G$ such that $1 = \mu(B_y, y) = \mu(F \setminus B_y, y')$ for all $y, y' \in C$ such that $y \neq y'$. Here $B_y = \{x \in F : (x, y) \in B\}$.

In Theorem 3.4, we present some sufficient conditions that guarantee continuous dependence of the posterior posterior probabilities of Borel sets on observations by using the generalized Bayes formula. Cotter et al [25, 26] have shown a closely related result which states that under certain conditions (including domination and Gaussian prior distribution), their version of the posterior distribution is Lipschitz continuous in finite-dimensional observations with respect to the Hellinger distance. Our proof relies on the Borel measurability of separately continuous functions – a result first obtained by Lebesgue and later generalized by Rudin [124]. The author was unable to find the proof of the measurability of separately continuous Souslin space-valued functions so the proof is included.

Note that even in a dominated case the posterior probabilities of Borel sets need not be continuous on any measurable linear subspace of full μ_Y -measure (see Section 5.4 and Remark 12 for an example). However, the posterior probabilities of Borel sets in a dominated case are always continuous on certain compact sets of nearly full measure (cf. Theorem 2.9). Unfortunately, in infinite-dimensional normed spaces the interior of any compact set is empty. The partial uniqueness of Theorem 2.7 is therefore not generic for infinite-dimensional normed spaces, unless there is a locally finite union of compact sets K_i such that $\cup_{i=1}^{\infty} K_i$ has full μ_Y -measure and the restriction of $\mu(U, \cdot)$ onto each K_i is continuous, which guarantees that $\mu(U, \cdot)$ is continuous on the whole $\cup_{i=1}^{\infty} K_i$. However, in Remark 3 we note that for any version of the posterior distribution there always exists some stronger topology on G that generates the same Borel sets, but makes the version continuous.

1.3. Case (iii): Posterior convergence. For computational reasons, the unknown X is often replaced with a finite-dimensional approximation X_n , where X_n is an F -valued random variable on the probability space (Ω, Σ, P) with finite-dimensional range. Instead of exploring the posterior distribution of X given $Y(\omega_0)$, we would like to explore the finite-dimensional posterior distribution of X_n given $Y_n(\omega_0) := L(X_n(\omega_0)) + \varepsilon(\omega_0)$, which is $\mu_n(U, Y_n(\omega_0)) = \mathbf{E}[1_U(X_n)|\sigma(Y_n)](\omega_0)$. As noted in [94, 96], the value $Y_n(\omega_0)$ is not given and the common procedure is to replace $Y_n(\omega_0)$ with $y = Y(\omega_0)$ in the expression of μ_n . We continue to call $\mu_n(\cdot, y)$ the posterior distribution of X_n , even though the replacement – strictly speaking – brings us out from the Bayesian world. One should note that continuity of the posterior distribution μ_n may additionally diminish the distortions in posterior opinions on X_n that are caused by replacing $Y_n(\omega_0)$ with a close observed value $Y(\omega_0)$. The question is then, do the posterior distributions $\mu_n(\cdot, y)$ on F (and the posterior means) converge when the approximations are refined?

Positive results for the convergence of either the posterior distributions $\mu_n(\cdot, y)$ or $\mu_n(\cdot, Y_n(\omega_0))$ have been given by Fitzpatrick [44] in the case of finite-dimensional observations of separable Banach space-valued unknowns, Lasanen [94] in the linear Gaussian case, Lassas and Siltanen [97] for the total variation prior on $C([0,1])$, Piiroinen [116] in the framework of statistical experiments for the Souslin space-valued random variables, Lassas et al [96] for certain Banach space-valued priors (including the Besov prior), Helin [63] for certain Hilbert space-valued priors (including an edge-preserving hierarchical prior), and Stuart [137] for a special form $f_n d\mu_0$ of the approximating posterior distributions, where $f_n \in L^1(\mu_0)$ for a Gaussian measure μ_0 .

The convergence in [44] is proved for the posterior probabilities of sets $P_n(U)$ where U is a Borel sets and the approximating operators P_n , where $n \in \mathbf{N}$ are continuous and converge to the identity on the Banach space. We note that since image of a Borel set under a continuous mapping in a Polish space is Souslin (see Theorem A.3.15 in [12]), the class of all sets $P_n(U)$, where $U \in \mathcal{F}$, is a subclass of all universally measurable sets. The convergence results in [63, 94, 96, 116] hold with respect to the weak convergence of measures i.e. $\lim_{n \rightarrow \infty} \mu_n(f) = \mu(f)$ for all continuous bounded real-valued functions f on F . The convergence results in [26, 137] are formulated for the Hellinger distance of the posterior distributions and the convergence results in [97] for weak convergence of the posterior distributions of the pointwise values of the unknown continuous function (i.e. the weak convergence in distribution). In [44, 97], the observations Y and the random variables $Y_n = L(X_n) + \varepsilon$ were assumed to have continuous probability densities with respect to the Lebesgue measure. The convergence results in [63, 94, 96, 97] are formulated for a linear forward theory L in the case of Gaussian noise. The converging posterior distributions in [94] are evaluated either at samples of $Y = L(X) + \varepsilon$ or the points $Y_n = L(X_n) + \varepsilon$, the converging posterior distributions in [116] are evaluated at the points $Y_n = L(X_n) + \varepsilon$, and the convergence results in [63, 96, 137] are formulated for fixed versions of the posterior distributions at given samples of $Y = L(X) + \varepsilon$.

The limit of the posterior distributions may not always be what one suspects. The famous example is the case of the so-called finite-dimensional total variation priors whose highly appreciated non-Gaussian posterior distributions converge weakly to a Gaussian distribution as the approximations are refined. Lassas and Siltanen [97] showed that this problem actually originates from the behavior of the prior distributions – the random variables $X_n(t)$ obeying total variation priors converge

to Gaussian limits when the discretizations are refined. This discovery, first conjectured by Markku Lehtinen, has changed the view on how Bayesian statistical inverse problems should be solved for infinite-dimensional unknowns – one should construct an infinite-dimensional prior distribution and check that the corresponding finite-dimensional posterior distributions converge to the right limit. Otherwise one risks the consistency of the prior knowledge and the consistency of the posterior distributions with respect to the increase in dimensionality. This guideline is followed in [63, 94, 96, 116].

Lassas et al [96] introduced a deterministic function on F , called the reconstructor R_g , that coincides a.s. with the conditional expectation of $g(X)$ given $Y^{-1}(\mathcal{F})$, where g is a measurable function having values in some separable Banach space. Lassas et al used a clever choice of their reconstructors R_U , $U \in \mathcal{F}$, which allowed them to state posterior convergence results for any given observation. The framework effectively transformed a question of originally probabilistic nature, the convergence of the conditional expectations $\mathbf{E}[1_U(X_n)|Y_n]$, into a question in analysis, the convergence of integrals. Moreover, it was possible to replace the samples of Y_n with samples of Y in the posterior distribution. The same technique is extensively used in the present work.

Unlike in [96], the convergence of the posterior distributions in Helin's work [63] is not based on approximating the prior random variables directly but on approximating the prior probability distributions in the weak topology. We adopt his viewpoint, since the posterior distribution depends on the prior random variable only through its distribution, assuming that the noise and the unknown are statistically independent (see Theorem 2.4 and Lemma 3.2).

Positive results for the convergence of posterior means for approximated unknowns have been obtained in the linear Gaussian case [94] (a.s. convergence in the Schwartz space $\mathcal{D}'([0,1])$), for the total variation prior on $C([0,1])$ [97] (for the pointwise-values), for exponentially integrable separable Banach space-valued priors [96] (in the norm topology), for uniformly discretized separable Hilbert space-valued priors with exponential weights [63] (in the norm topology for all exponentially bounded functions), for polynomially bounded functions and those posterior approximations that have the form $f_n d\mu_0$, where $f_n \in L^1(\mu_0)$ for some Gaussian measure μ_0 [26, 137].

In Section 4, some results in [25, 63, 96, 116, 137] that concern the weak convergence of posterior distributions and convergence of the posterior means are extended in several directions.

Firstly, we allow prior distributions to be probability measures on a locally convex Souslin topological vector space F , whereas Lassas et al [96] and Helin [63] applied the generalized Bayes formula for separable Banach and Hilbert space-valued unknowns, respectively. Posterior distributions in locally convex Souslin (not metrizable) topological vector spaces have been considered before in the special case of Gaussian distribution-valued random variables in [94, 100] and in the general abstract case of Souslin space-valued random variables only in [116]. The first part of Section 2 is therefore devoted to the basics of the abstract statistical inverse theory in locally convex Souslin topological vector spaces. Unlike in the work of Piiroinen [116], where general Souslin space-valued random variables were first studied in statistical inverse problems, we use the generalized Bayes formula for the proofs. We also derive the generalized Bayes formula from the equation $Y = L(X) + \varepsilon$, which supplements also the formulation presented in [25, 26, 137]

where the starting point is a given form of the posterior distribution. This makes it easier to recognize situations where the generalized Bayes formula holds.

In this work, the use of locally convex Souslin topological vector spaces is mostly motivated by the fact that in Souslin spaces the Borel σ -algebras are regular enough for the existence of regular conditional measures [13]. Moreover, the class of such spaces contains many useful spaces, like complete separable metric vector spaces and spaces of (Schwartz) distributions [129]. We remark that the distribution spaces are sometimes preferred since the convergence of the characteristic functions of measures implies the weak convergence of measures for them (this fact is shown e.g. in [13] and used in [94]). We require G to be a topological vector space since Y is defined as the sum of two G -valued random variables. However, it is well-known that the sum of two random variables is not always a random variable in arbitrary topological vector spaces. In Lemma 3.1, we check that Y is indeed a random variable because the sample space G is a Souslin space (the fact is known but the author was unable to find a reference for the proof in the literature). We require G and F to be locally convex topological vector spaces since locally convex spaces have rich enough dual spaces that for example allow the use of characteristic functions in the identification of measures. In Remark 1, we note that a locally convex Souslin sample space of Y is allowed to be mis-specified by a continuous linear injection without altering the posterior distributions. This holds for any statistical inverse problem, not just for those admitting the representation (3).

The main difference of the present Theorem 4.4 to Theorem 4.8 in [116] is that we do not require the conditioning σ -algebras $Y_n^{-1}(\mathcal{G})$ to be increasing. This is a significant difference as it allows more general approximation schemes. On the other hand, the present approach utilizes the generalized Bayes formula, which was not needed in [116]. Hence, the results in [116] are valid for many noise models that are not covered by our assumptions, like additive undominated noise, multiplicative noise, or noise that is statistically dependent on the unknown (we assume that the noise is statistically independent from X and all of its approximations X_n). Another difference from Theorem 4.8 in [116] is that we work with one fixed sample y of $Y = L(X) + \varepsilon$ whereas in [116] it is assumed that we have a sequence $\{y_n\}$ consisting of samples of the random variables $Y_n = L(X_n) + \varepsilon$, which is a drawback when one considers realistic observations. The reason for this is that in [116] the convergence is shown for the conditional expectations $\mathbf{E}[f(X_n)|\sigma(Y_n)](\omega)$ (which are equivalent to $\mu_n(f, Y_n(\omega))$) for P -almost every $\omega \in \Omega$). However, in Lemma 4.15 in [116], it is explained how Theorem 4.8 can be used in certain cases where the observation $Y = L(X) + \varepsilon$ is given. Namely, the prior distribution of the Souslin space-valued random variable X is assumed to be concentrated on a separable Hilbert space H and its approximations are of the form $X_n = P_n X$, where the operators P_n are some finite-dimensional linear operators on H that converge to the identity at every $x \in H$. Assuming that the range of the linear operator L is some separable Hilbert space \tilde{H} and the noise $\varepsilon \in \tilde{H}$ with probability one, Piironen constructed certain projection operators R_n , and showed that the sequence of the posterior distributions of X_n given $R_n(LX + \varepsilon)$ converges weakly to the posterior distribution of X given $LX + \varepsilon$ as $n \rightarrow \infty$. The result of Piironen shows, remarkably, that in some cases less data is adequate – and easier to manage – than full data. We remark that the required assumptions exclude injective compact linear operators of infinite rank as L . Indeed, if $L : H \rightarrow \tilde{H}$ is any compact linear bijection, then its inverse operator is bounded by the open mapping theorem. Considering $LL^{-1} = I$ on \tilde{H} and $L^{-1}L = I$

on H , we see that L can not be compact unless the Hilbert spaces H and \tilde{H} are finite-dimensional. On the other hand, compactness of L is a typical feature leading to the ill-posedness of the inverse problem. But under the assumptions of Lemma 4.15 in [116], the convergence result of Piironen is stronger than ours in the sense that it allows any noise model with the property $\varepsilon \in \tilde{H}$ with probability 1. However, the compactness of L is not a restriction for the present convergence results. The weak convergence of posterior distributions in the linear Gaussian case in [94] is not fully covered by the present results since some of the cases appearing in [94] are not μ_X -a.s. dominated.

Secondly, we allow a wider class of noise models than the Gaussian models applied in [63, 96]. Theorem 4.4 gives a positive answer to the weak convergence of the posterior distributions in locally convex Souslin topological vector spaces, when the translations of the noise distribution by $L(x)$ are μ_X - and μ_{X_n} -a.s. dominated and the corresponding Radon-Nikodym densities satisfy certain measurability and uniform integrability conditions. Examples of suitable noise models are given in Section 5, which include the well-known cases of finite-dimensional noise and Gaussian infinite-dimensional noise but also four novel models such as spherically invariant noise and periodic signals in decomposable noise.

Theorem 4.6 extends the convergence of the posterior means in [63, 96] for more general noise models, and relaxes slightly the integrability properties imposed on the approximations of the unknown in [63]. Theorem 4.6 extends also assumptions in Theorem 4.10 of [137] for more general approximations of the prior distributions (but the mode of convergence is different). Some conditions, which imply the posterior convergence of continuous linear functionals for weakly converging prior distributions, are presented in Theorem 4.5. The mentioned conditions are indebted to the well-known criteria for the convergence of integrals with respect to measures that converge weakly.

Thirdly, we consider stronger modes of convergence for posterior distributions than the weak convergence considered in [63, 96]. In Theorem 4.7, we give sufficient conditions under which the posterior distributions inherit also the setwise convergence or the convergence in variation of the approximated prior distributions. Recently, Stuart has established (see Theorem 4.6 in [137]) an estimate for the speed of convergence in Hellinger distance of the posterior distributions for the approximated posterior distributions of the restricted form $\mu_n(dx, y) = f_n(x, y)d\mu_0(x)$, where $f_n(\cdot, y) \in L^1(\mu_0)$ and μ_0 is a Gaussian measure on a Banach space F . In the present Theorem 4.7, the approximated posterior need not be absolutely continuous with respect to a Gaussian measure, and the approximations X_n need not be measurable functions of X .

Moreover, we allow the direct theory L to be nonlinear (as in the less general cases in [25, 26, 137]), which is a minor modification of the linear case in [63, 96], but indicates that nonlinearity does not necessarily complicate the mathematical convergence, although the exploration of the posterior distribution becomes more difficult. The result is not surprising since nonlinearities are frequently handled in the stochastic filtering problems [78, 111], which have connections to the statistical inverse problems. Throughout the paper, we consider continuous forward mappings $L : F \rightarrow G$, although the existence of posterior distributions requires only their measurability. However, the continuity of L is utilized in the main results of the

present paper on the convergence of the posterior distributions and on the well-posedness of the Bayesian statistical inverse problem. It is also consistent with the usual description of the deterministic inverse problems where continuity holds.

Unlike in [63, 94, 96, 116], the case of approximated observations is not studied (nor reviewed) in the present work. By focusing on the approximated unknowns, we hope to single out their essential properties. Section 6 contains some examples of prior approximations.

Notations: When G is a topological vector space, we denote with G' its topological dual space. If m is a measure on G , we sometimes denote $m(f) := \int_G f(x) dm(x)$. If $Z : \Omega \rightarrow G$ is a random variable, its image measure $P \circ Z^{-1}$ on G is denoted with μ_Z . A Borel measure and its Lebesgue's completion are denoted with the same symbol.

1.4. A literature review. Statistical inverse theory became a popular method for solving geophysical problems in 1980's [138, 139], and has since spread into many other fields (see [75, 137]). In this short review, we focus on general theoretical developments that lead to the modern description of statistical inverse theory. A more problem-oriented review of infinite-dimensional Bayesian statistical inverse problems can be found in [137], and reviews of statistically oriented inverse problems can be found in [19, 141]. A good reference to the computational aspects of finite-dimensional statistical inverse problems is [75], to Bayesian statistics [45, 122, 128], and to measure theory [13].

The statistical background of the statistical inverse theory belongs to the field of nonparametric Bayesian inference. Nonparametric statistics is concerned with making inferences about infinite-dimensional unknowns whereas parametric statistics studies finite-dimensional unknowns [7]. The function-valued prior models in statistical inverse problems are therefore well within the scope of nonparametric statistics. We briefly review Bayesian nonparametric statistics and clarify its relations to statistical inverse problems.

Important nonparametric problems are the density estimation problem and the regression problem [106]. These two problems have guided the modern development of Bayesian nonparametric statistics.

In the density estimation problem, the observations are i.i.d. samples obeying some unknown probability distribution that has a density function f (usually on \mathbf{R}), and the objective is to estimate the density function f . This problem is not directly related to our statistical inverse problem but is connected to the general development of the research field. It should be mentioned that Wolpert et al [159, 160] have described a semidiscrete Fredholm integral equation of the first kind as a Bayesian density estimation problem. On the other hand, the positron emission tomography (PET) imaging is an inverse problem that is usually described as a special density estimation problem where only indirect samples are available [72]. Hence, certain inverse problems lead to density estimation problems.

In the regression problem, the observations are of the type

$$y_i = K(x_i) + \varepsilon_i.$$

where $x_i \in \mathbf{R}^n$, $i = 1, \dots, n$, and the noise terms ε_i are typically independent and identically distributed. The objective is to estimate the unknown function K . This problem has connections to statistical inverse problems. For example, if the realizations of X and $Y = L(X) + \varepsilon$ are functions on \mathbf{R} and L is the identity mapping, then X is identified as K .

One difference between the density estimation problem and the regression problem is the nature of given samples. In the regression problem, a single sample can also be infinite-dimensional, at least in theory. When the noise is Gaussian, such infinite-dimensional observation models are often called white noise models (see the short review in [164]).

The main questions in Bayesian nonparametrics have been the construction of the prior models, the utilization of the posterior distribution, and the consistency of the posterior distributions.

1.4.1. Prior models. The problem of finding good infinite-dimensional prior models has a long history. An early application of a function-valued prior model was carried out in 1896 by Poincaré [118], who applied a random series of the type $X(t) = \sum_{i=1}^{\infty} X_i t^i$ in a regression problem on $[0, 1]$. He assumed independent normal distributions on coefficients X_i and calculated the posterior mean estimate on the basis of the given values $y_i = X(t_i)$, $i = 1, \dots, n$. In Section V of Chapter 11 in [117] Poincaré discussed, in his visionary manner, the noisy regression problem. He proposed that the smoothness of the regression curve follows from the prior information of the unknown curve described in the form of probability distributions. In 1950, Grenander [58] applied a Gaussian process prior in a linear regression problem with additive Gaussian process noise. In 1957-58 Whittle [153, 154] discussed prior information on the smoothness of the unknown in certain density estimation problems, and later Kimeldorf and Wahba [82] clarified the relations between smoothing and Gaussian prior models. Nowadays, regularity of functions is one of the most important guidelines in constructing infinite-dimensional prior models in statistical inverse problems. This follows from the fact that the priors in statistical inverse problems have two objectives. They express the prior beliefs about the unknown and are countermeasures against the ill-posedness of the deterministic inverse problem.

In general, the knowledge on infinite-dimensional random variables (and on their distributions) started to increase after Wiener published his construction of the Brownian motion in the beginning of 1920's [156]. A decade later, Kolmogorov [84] introduced a constructive method for defining general infinite-dimensional random variables in the abstract setting. His method suited well for countably many random variables, but Doob noticed that the constructed σ -algebra was somewhat limited: for continuous parameter processes certain interesting sets, such as the set of all continuous functions, were not measurable with respect to the constructed σ -algebra. Doob's remedy was the careful definition of the continuous-parameter stochastic processes in 1937 [36]. The theory of stochastic processes Doob's definition of the separable stochastic processes provides the tools but not immediate answers for certain questions in statistical inverse problems. Namely, can the stochastic process be interpreted as a function-valued random variable that has values in some nice function space? The question is quite relevant since the direct theory is a mapping between two function spaces. One can e.g. apply Kolmogorov's continuity theorem (proven by Kolmogorov in 1934, see [134]). Another approach is to directly define probability measures on function spaces. Jessen [69] carried out integration on infinite-dimensional dimensional torus equipped with the coordinate-wise convergence. M. Fréchet initiated the study of random variables in metric spaces (see [50]). His emphasis was on different modes of convergence and typical values of random variables, like the mean and the median. Significant contributions to the theory of probability measures on topological spaces were given by Alexandrov and Prohorov (see [149]). Later developments can be found in the books of Bogachev

[12, 13], Gelfand and Vilenkin [53], Gihman and Skorokhod [55], Kahane [73], Kuo [91], Ledoux and Talagrand [98], Schwartz [129], Vakhania [147], Xia [161], and Yamasaki [163]. Typical points discussed in these books are the existence of measures, invariance properties of measures, topological supports, the equivalence and the equality of measures, the convergence of random series and the convergence of measures, all relevant properties for prior distributions. The existence of measures is often based on the Bochner-Minlos theorem that gives conditions for the one-to-one correspondence between measures μ and their characteristic functions $L(\phi) = \int e^{i\langle x, \phi \rangle} d\mu(x)$ on certain spaces. From the point of view of statistical inverse problems, it is unfortunate that direct connections between the characteristic function and the included prior information are not known. Therefore, it is no wonder that popular prior models have been described by other means, for example with infinite product measures and random series expansions. The works of Karhunen [80] in 1940's on a series expansions of Gaussian random variables, nowadays known as the Karhunen-Loève expansion, are in this sense important. The Karhunen-Loève expansion was first used in 1950's and 1960's for expanding infinite-dimensional data [30, 58, 81], which made the Bayesian method of conditional mean estimation and the non-Bayesian method of likelihood ratio testing tractable. It was later adopted to describing infinite-dimensional unknowns (see for example [27]), but its main application has been in providing finite-dimensional approximations of Gaussian random variables. At present, other orthogonal expansions of Gaussian random variables are available [12]. The pioneering work of Mandelbaum [105] from 1984 on linear Gaussian statistical inverse problems relies on such series expansions of the Gaussian random variables. Other works on Gaussian priors in statistical inverse problems are [46, 94, 100, 102, 133].

In 1963, Freedman introduced the class of tail-free priors for the density estimation problem [51]. In 1970's the density estimation and the regression problem evolved further in different directions. Wahba et al [82, 150] took the approach with smoothing splines and Gaussian random series in the regression problem, and Ferguson [43] constructed Dirichlet process priors, which are certain random measure-valued unknowns, for the density estimation problem. In the case of Dirichlet processes, the space of the unknowns is the space of all probability measures on the fixed measure space equipped with the Borel σ -algebra with respect to the weak topology of measures. The Dirichlet process priors have similar properties in the density estimation problem as Gaussian priors have in the linear statistical inverse problems. Namely, the posterior distribution is the distribution of another Dirichlet process with updated parameters. In the both cases, the calculations of the posterior distribution are based on similar elements, which are the properties of the finite-dimensional distributions and the properties of the martingales [43, 105]. The Dirichlet process priors were generalized later to mixtures of Dirichlet processes (see [41]). Summaries of the prior distributions applied in modern density estimation problems can be found in [21, 151].

In 1990, Steinberg [135] suggested a prior model defined as a random series in which Hermite polynomials were multiplied by either improper or Gaussian coefficients. During 1998-2000, Abramovich et al [1, 2, 3] suggested random wavelet expansions in Besov spaces, with hierarchical coefficients whose hyperparameters guaranteed the sparseness of the expansions, as priors for the regression problem. In 1990's also mixtures of Gaussian measures were suggested as priors for the regression problem (see [164]). Recently, Lassas et al [96] and Helin [63] constructed

non-Gaussian edge-preserving priors suitable for statistical inverse problems. Besov space priors introduced by Lassas et al are defined with random wavelet expansions and Helin's hierarchical prior distributions as mixtures of Gaussian measures. In 2010, Stuart [137] applied prior distributions of the type $f(x)\mu(dx)$, where $f \in L^1(\mu)$ and μ is a Gaussian measure. In the abstract setting, the statistical inverse theory was applied for unknowns described as Souslin space-valued random variables in [116].

It should be mentioned that some combinations of prior informations do not have faithful probabilistic descriptions. In 1987, Backus [5] pointed out that hard constraints, such as the boundedness of an infinite-dimensional random variable X in norm, can lead to troubles if one assumes also isotropy. A well-known example is a Gaussian random variable X that is invariant with respect to rotations (e.g. orthogonal transformations) on an infinite-dimensional separable Hilbert space H but satisfies $\|X\|_H = \infty$ with probability one [12].

1.4.2. Utilization of the posterior distribution. We first look at the history of infinite-dimensional posterior distributions in nonparametric statistics and in statistical inverse problems.

In Bayesian nonparametrics, the both problems, the density estimation and the regression problem, are solved with conditional probability measures. This part of the solution mechanism is exactly the same as in statistical inverse theory. In 1930's, the rigorous definition of the conditional expectation by Kolmogorov [84] made it possible to define conditional probability measures in the abstract infinite-dimensional setting but it was soon noted that such conditioning did not always produce a probability measure. The works of Doob [37] and Dieudonné [34, 35] lead to the definition of a regular conditional probability, which is a random probability measure with probability one. The existence of regular versions of all conditional probabilities was verified by applying certain properties of the space of the unknowns in the works of Rohlin [123], Jiřina [70, 71], and Sazonov [127]. Nowadays, one either checks the properties of the space of the unknowns (as in [116]) or checks always the regularity of the acquired conditional measure for the chosen prior distribution (as in [43]). The former is used in theoretical studies for avoiding pathological cases [63, 94, 96, 116] while the latter is convenient in practical solutions where a fixed version is needed [25, 137]. We remark that the non-existence of a regular version is known only for some conditional measures in exemplifying cases (see [13, 120]).

A major step for the statistical inference for stochastic processes was the emergence of the so-called filtering problems in 1940's by Wiener [155], Kolmogorov [85] and Krein [87, 88]. Especially, Wiener's [155] straightforward method of solution (by ergodicity and least squares estimation) encouraged others to take later further steps towards the Bayesian nonparametric approach [48, 58, 154]. A good review on developments in the filtering theory is [74]. An interesting work in the filtering theory is [9], where it is shown that the solution of the filtering problem depends continuously on the distribution of the unknown. A nice collection of nonlinear filtering problems with Gaussian noise can be found in [104].

The first deliberate unions of inverse problems and Bayesian statistics were seen in 1960's in the form of statistical regularization i.e. minimum mean squared error estimation for the Gaussian linear inverse problem

$$(5) \quad Y = LX + \varepsilon$$

with the finite-dimensional unknown X and the finite-dimensional observation Y . That is, one pursues after the estimator $\hat{X}(Y)$ that minimizes $\mathbf{E}[\|\hat{X} - X\|^2]$ (i.e. the conditional mean). In 1961, motivated by Wiener's filtering theory, Foster [48] presented a solution to the estimation problem (5). Other motivation for the Gaussian approach arose from the regularization method of Philips [115] generalized later by Twomey for Fredholm integral equations of the first kind [144] and from the Tikhonov regularization method. During 1967-71 Turchin et al (see [143] and references therein), independently with Strand and Westwater [136], replaced the regularization method by a statistical framework that utilized a Gaussian prior distribution. The approach lead to Franklin's infinite-dimensional description [49] of the minimum mean squared error estimator of a Hilbert space-valued Gaussian unknown whose linear observations were corrupted by an additive Gaussian white noise. The connection between [49] and regularization methods in reproducing Hilbert spaces were studied by Prenter and Vogel [119]. The first work that contained the existence of regular conditional probabilities and an explicit formula for the posterior distribution in a linear infinite-dimensional inverse problem was the seminal paper of Mandelbaum [105] on Hilbert space-valued Gaussian random variables. The value of the result for inverse problems was first recognized by Lehtinen et al [100] who generalized it for the Gaussian (Schwartz) distribution-valued random variables. This work of Lehtinen et al can be considered as the starting point of the infinite-dimensional Bayesian inverse problems. The case of Banach space-valued Gaussian random variables was later considered by Luschgy [102]. In these works, the expression of the posterior mean is obtained by using the equivalence between statistical independence of Gaussian random variables and their orthogonality in $L^2(P)$. The key factor is the orthogonal random series expansion of the Gaussian observation – a method used by Grenander [58], and even by Poincaré [118]. Cox [27] applied Gaussian separable Banach space-valued unknowns in a linear regression problem with additive Gaussian noise. The approach of Cox differs from that of Mandelbaum since it uses the generalized Bayes formula rather than the special properties of Gaussian random variables (see Proposition 2.1 in [27]). An abstract formulation of Bayesian statistical inverse problems for Souslin space-valued random variables was given by Piiroinen [116], who only required the observation and the unknown to be Souslin space-valued random variables, thus allowing nonlinear direct problems and more complicated noise terms.

Little is known about the form of posterior distributions in infinite-dimensional statistical inverse problems outside the Gaussian linear case [94, 100, 102, 105, 133] and the dominated case with Gaussian noise [63, 96, 164]. When F and G are complete separable metric spaces, a result of Macci [103] tells that the Lebesgue decomposition of the posterior distribution with respect to the prior distribution contains a nontrivial singular part in undominated cases. Namely, the Lebesgue decomposition of the posterior distribution with respect to the prior distribution is of the form $\mu(\cdot, y) = \mu^{(ac)}(\cdot, y) + \mu^{(s)}(\cdot, y)$, where the absolutely continuous part $\mu^{(ac)}(\cdot, y)$ is determined by the absolutely continuous part of $\mu_{Y|X}(\cdot, x)$ with respect to μ_Y through the equations

$$\mu^{(ac)}(U, y) = \int_U \frac{d\mu_{Y|X}^{(ac)}(\cdot, x)}{d\mu_Y}(y) d\mu_X(x),$$

where $U \in \mathcal{F}$. Moreover, the singular part $\mu^{(s)}(\cdot, y)$ is determined by the singular part of $\mu_{Y|X}(\cdot, x)$ with respect to μ_Y through the equations

$$\mu^{(s)}(U, y) = \frac{d\left(\int_U \mu_{Y|X}^{(s)}(\cdot, x) d\mu_X\right)}{d\mu_Y}(y), \quad U \in \mathcal{F},$$

from which one chooses a regular version. We remark that in such undominated cases, one may expect to meet some surprises. The posterior distribution presents then some things that seemed to be *a priori* impossible.

The extraction of information from the posterior distribution involves decision theory, including point estimation and hypothesis testing (for the general description of the Bayesian decision theory, see [128]). A decision theoretic view towards Bayesian inversion is given in [40], where one performs the estimation of the (separable Banach space-valued) unknown by first fixing the prior distribution of the unknown X and choosing a loss function $\ell : F \times F \rightarrow \mathbf{R}$ that penalizes the inaccuracies in the estimates of the unknown, and then choosing the so-called Bayes estimator $\hat{X} : G \rightarrow F$, which is a deterministic function that gives the smallest averaged loss $\mathbf{E}[\ell(\hat{X}(Y), X)]$. This is equivalent to taking as each $\hat{X}(Y(\omega))$ the value d that minimizes the posterior expected loss $\mathbf{E}[\ell(d, X)|\sigma(Y)](\omega)$.

Common point estimators in finite-dimensional statistical inverse problems are the maximum a posteriori (MAP) estimator and the conditional mean (CM) estimator (i.e. the posterior mean) [75]. The CM estimator $\hat{X}(Y) = \mathbf{E}[X|\sigma(Y)]$ minimizes the posterior risk for the squared error loss function $\ell(x', x) = |x' - x|^2$ (when $X : \Omega \rightarrow \mathbf{R}^n$ is suitably integrable) [75]. Conditional means have appeared also in the framework of infinite-dimensional statistical inverse problems [63, 94, 96, 97, 100, 102, 105]. However, the decision-theoretic justification is often neglected, and the conditional mean is reported just as a typical value of the posterior distribution. Other notions of typical values for distributions on separable metric spaces were considered by Frechét [50].

The mean of a locally convex Hausdorff topological vector space-valued random variable can arise from different definitions, depending on the space in question. In general, the (weak) mean of a locally convex Hausdorff topological vector space-valued random variable X is a vector $m \in F''$ (or more generally, m in the algebraic dual space of F') such that $\langle m, \phi \rangle_{F'', F'} = \mathbf{E}[\langle X, \phi \rangle_{F, F'}]$ for all $\phi \in F'$ P -a.s. (see [12]). Such notion of vector-valued integration was developed by Pettis [114] in 1933 for reflexive separable Banach spaces F . Gelfand used a similar definition for distribution-valued random variables (see [53]). The Pettis-Gelfand integral was generalized for quasi-complete Souslin space-valued functions by Thomas [142]. For Banach-space valued random variables having integrable norm, a mean can be defined also as the Bochner integral $m = \int_F x d\mu_X(x)$, introduced in early 1930's by Bochner (see [33] and references therein).

When the posterior distribution $\mu(\cdot, y)$ is known for a given sample y of Y , the (weak) conditional mean $m \in F''$ is a vector that satisfies $\langle m, \phi \rangle_{F'', F'} = \int \langle x, \phi \rangle_{F, F'} \mu(dx, y)$ for all $\phi \in F'$. When F is a separable reflexive Banach space and $\|X\|$ is integrable, the same posterior mean can also be defined as the Bochner integral (see Proposition V.2.5 in [108]).

We remark that the weak posterior mean $\mathbf{E}[X|\sigma(Y)]$ is a Bayes estimator in a weak sense i.e. it gives the smallest averaged loss for the family of loss functions $\ell_\phi(x, x') = |\langle x - x', \phi \rangle|^2$, where $\phi \in F'$. Franklin [49] used such requirement, when he defined the best linear estimator in a Gaussian linear inverse problem in 1970. An earlier approach to the best linear estimator in function-valued Gaussian case

was given by Grenander in 1950 (see Chapter 6 in [58]). He considered a Gaussian linear regression problem and identified the best linear estimator (with respect to the pointwise squared error loss $\ell(X(t), \hat{X}(t)) = |X(t) - \hat{X}(t)|^2$) with the posterior mean. Grenander used infinite-dimensional observations, but he made simultaneous inferences on only finitely many values of the unknown function. Moreover, he required, but not proved, the regularity of the conditional probabilities. In this sense, his approach to the posterior means was still far from the description of Mandelbaum from 1984 [105]. Remark, that the technique of estimating the value of $X(t)$ on the basis of infinitely many observations is still the standard in the modern filtering theory [111]. Also in the Bayesian density estimation, the estimation is sometimes carried out in either in the form $\hat{X}(t) = \mathbf{E}[X(t)|Y_1, \dots, Y_n]$, where X is the unknown probability density function on, say $[0, 1]$ (see [154]), or in the form $\hat{X}(U) = \mathbf{E}[X(U)|Y_1, \dots, Y_n]$, where the sets $U \subseteq [0, 1]$ are Borel set and X is an unknown random probability measure (see [106] and Proposition 4.2.1 in [54]). The density estimator \hat{X} is a Bayes estimator with respect to the squared error loss function for each t or for each Borel set U , respectively. An other option is to use a weighted L^2 -loss function $\ell(\hat{X}, X) = \int_0^\infty |\hat{X}(t) - X(t)|^2 dw(t)$ [43]. The two estimators coincide when X is suitably integrable.

In the works of Mandelbaum [105] and Luschgy [102], the space F is a Hilbert or Banach space, and the posterior mean is defined as a Bochner integral. However, the emphasis is on the Gaussian nature of the prior, and the posterior mean is calculated as $\mathbf{E}[\sum_{i=1}^\infty X_i e_i | \sigma(Y)] = \sum_{i=1}^\infty \mathbf{E}[X_i | \sigma(Y)] e_i$. Similar approach appears in [94, 100] for the distribution space, where the posterior mean is defined in the weak sense. The weak definition of the posterior mean is used also in [97] for the space $C([0, 1])$. In [63, 96], the conditional mean of a separable Banach space-valued random variable is defined as a Bochner integral with respect to the posterior distribution. Before Luschgy, Krug [89] determined the posterior mean of a separable Banach space-valued Gaussian unknown in a linear Gaussian case, but he assumed that the given observation was finite-dimensional.

We remark that when F is a Hilbert space, one can take $\ell(x', x) = \|x - x'\|_F^2$ as the loss function that gives the CM estimator. As in the finite-dimensional case, the main point is that

$$\begin{aligned} \mathbf{E}[\|\hat{X}(Y) - X\|_F^2] &= \mathbf{E}[\|\hat{X}(Y) - E[X|\sigma(Y)]\|_F^2] + \\ &\mathbf{E}[(\hat{X}(Y) - E[X|\sigma(Y)], E[X|\sigma(Y)] - X)_F] + \mathbf{E}[\|X\|_F^2], \end{aligned}$$

and the additional difficulty is just in checking that $\mathbf{E}[(f, X)|\sigma(Y)] = (f, \mathbf{E}[X|\sigma(Y)])$. Such loss functions have been used in the regression problem for the Gaussian mixture priors when $F = L^2([-1, 1])$ [164]. Instead of an L^2 -loss function, Abramovich et al [1] used an L^1 -loss function in a regression problem for a discretized Besov space-valued unknown. We note that a common approach in the regression problem is to present only the Bayes estimates instead of the whole posterior distribution.

Luschgy [102] made an (unproven) remark that for Gaussian posterior distributions the conditional mean is the Bayes estimator for every symmetric quasi-convex (measurable) loss function $\ell(x, x') = \ell(x - x')$. A proof can be found in [15], where it is derived from the Anderson property of Gaussian measures (for the property, see [101]).

In finite-dimensional spaces, the MAP estimator can be interpreted as a limit of Bayes estimators for the 0-1-valued losses $\ell_\epsilon(x', x) = 1_{F \setminus \overline{B}(x, \epsilon)}(x')$, where $\epsilon \rightarrow 0$

[122]. Here $\overline{B}(x, \epsilon)$ is the closed ball in F that is centered at x and has radius ϵ . Lassas and Siltanen [97] showed that MAP estimates can behave inconsistently as dimensionality of the unknown increases, even though the posterior distributions converge at the same time. In their example, the MAP estimates actually vanish at the limit, regardless of the given observation. Similar result is proved in [64] for a hierarchical edge-preserving prior. The MAP and CM estimates coincide for the finite-dimensional Gaussian priors, and numerical results demonstrate that they can practically coincide for the finite-dimensional approximations of Besov-priors [83]. Cotter et al [25] discussed MAP estimation in the context of infinite-dimensional Bayesian problems. They showed that there exists a minimizer for a penalized log-likelihood function, which has similar form as in the case of finite-dimensional Gaussian unknown. However, the conditions that would relate the penalized log-likelihood function to any posterior density were omitted in [25], which leaves open the question what connections the minimizer has to the infinite-dimensional posterior distributions. Recalling the result of Lassas and Siltanen [97] arises at least some caution. Another attempt towards MAP estimation with infinite-dimensional Gaussian priors is given by Hegland [62]. Unfortunately, the proof of Proposition 1 in [62] is not rigorous, as it involves subtraction of two numbers that are infinitely large with probability 1 (i.e. the Cameron-Martin norms of arbitrary vectors in the space of the unknowns).

In infinite-dimensional statistical inverse problems the hypothesis testing has been largely neglected, although several interesting question could be raised. For example, Fitzpatrick [44] has made an initiative on testing if the evidence supports the homogeneity of the unknown diffusion coefficient. Hypothesis testing was proposed also for some nonparametric statistical inverse problems in [11] within the classical framework. However, it was pointed out in [11] that the problems can be similarly handled also by the (finite-dimensional) Bayesian methods but this remark is not elaborated further.

Another approach to exploiting posterior distributions was given by Piironen [116]. He interpret the posterior distributions as statistical measurements, which allowed comparisons of information contents of different posterior distributions. The result is especially useful in experimental design [99].

1.4.3. Posterior consistency. The consistency of the posterior distributions (with respect to repeated independent observations) is closely connected to the uniqueness of the deterministic inverse problem of determining x from $y = L(x)$. The pioneering work of Doob [38] on martingales touched the question of consistency of the posterior distributions. Doob's results imply that under model identifiability (i.e. the measures $\mu_{\epsilon+L(x)}$ are different for different $x \in F$) the posterior distributions would concentrate (in the weak topology of measures) on the true unknown x_0 $\mu_{L(x_0)+\epsilon}$ -almost surely for μ_X -a.e. x_0 when infinitely many i.i.d. observations would be available. The consistency of the posterior distribution is an important topic because it shows that enough data will guide a Bayesian scientist almost surely to the true answer. The words μ_X -a.s. made Doob's approach slightly impractical as they left open the frequentist case where the observations are not samples of $L(X)+\epsilon$ but samples of $L(x)+\epsilon$ for some fixed x . Freedman [51] demonstrated that inconsistency could hold on topologically large sets. The problem was approached by Schwartz [130] who described a set of unknowns x for which consistency holds $\mu_{\epsilon+L(x)}$ -almost everywhere under some decision theoretic conditions and domination (i.e. all measures $\{\mu_{\epsilon+L(x)} : x \in F\}$ are assumed to be absolutely continuous with respect to

some common σ -finite measure). The required property is the positive prior probability of all Kullback-Leibler neighborhoods of the unknown x . Consistency has been studied also in other topologies, beside of the weak topology. Barron et al [6] proved consistency in Hellinger distance. Summaries of consistency result in density estimation can be found in [31, 32, 162]. The case of Gaussian regression has been studied in [148], where certain probabilities are shown to converge.

Consistency issues in inverse problems are discussed in [40]. For our statistical inverse problem, the consistency corresponds to observing one sample of $Y = L(X) + \frac{1}{\sqrt{n}}\varepsilon$, where n represents the number of i.i.d. observations of $L(x) + \varepsilon$. The works of Hofinger and Pikkarainen [67, 68], and Neubauer and Pikkarainen [107] on finite-dimensional Gaussian statistical inverse problems concern the question of posterior consistency. They studied the convergence of posterior distributions and the posterior means in linear Gaussian inverse problems for finite-dimensional random variables as the variance of the noise decreases. In particular, it was shown in [67] that the posterior distributions given observed values $\tilde{Y}_{\delta_n} = Lx + \delta_n\varepsilon(\omega)$ of $Y_{\delta_n} = LX + \delta_n\varepsilon$ for a sequence $\delta_n \rightarrow 0$, converge to the point mass on the true value x in the Ky Fan metric, assuming that also the prior distribution are modified appropriately. Hofinger and Pikkarainen [107] studied posterior convergence rates for finite-dimensional approximations of Hilbert-space-valued random variables when the approximation level increases in certain manner as the noise level δ_n approaches to zero. However, the convergence was shown only for unknowns in an *a priori* zero measurable set (the Cameron-Martin space of the prior distribution). Also Florens and Simoni [46] studied the posterior consistency for the infinite-dimensional linear Gaussian inverse problems when the variance of the noise diminishes. They were able to show the posterior consistency if the posterior measures with respect to the weak topology (and give estimates for the speed of convergence of the posterior means) by assuming that the direct theory is regular enough and the prior distribution depends suitably on the noise level.

Another convergence topic that has received more attention in statistical inverse problems is the posterior convergence for approximated unknowns and observations [63, 94, 96, 97, 116, 137]. This case has been discussed above in the introduction.

Almost all known convergence results for posterior distributions [4, 63, 94, 95, 96, 97, 137] are based on the known form of the posterior distribution. There are also some measure-theoretic approaches for convergence of conditional expectations. The results of Gänssler and Pfanzagl [52] showed that the conditional expectations $E[1_U(X_n)|Y]$ converge when the joint distributions of the observation and the approximated unknowns are dominated by some σ -finite measure and the corresponding Radon-Nikodym densities converge almost everywhere. Furthermore, they also showed that there exists a regular version for which the convergence holds almost surely in variation. Here one should pay attention to the fact that the conditioning σ -algebra does not depend on n which is not satisfactory from the point of view of numerical solutions of statistical inverse problems. Landers et al [93] generalized this result for monotonic sequences of conditioning σ -algebras $\sigma(Y_n)$. It should be noted that in statistical inverse problems, the σ -algebras $\sigma(L(X_n) + \varepsilon)$ are usually not increasing. Krikkeberg [86] proved a martingale type convergence theorem for not necessarily monotonic σ -algebras, but his conditions seem to be too abstract for the statistical inverse problems in the present form. A reformulation of his conditions in terms of random variables (X_n, Y_n) would give valuable information on the almost sure posterior convergence in the general undominated case. Goggin [56] and

Crimaldi et al [28, 29] studied conditions under which the convergence of (X_n, Y_n) to (X, Y) implies the convergence of the conditional expectations $\mathbf{E}[f(X_n)|Y_n]$ to $\mathbf{E}[f(X)|Y]$ in distribution or in probability. Their results are not satisfactory for statistical inverse problems, since they do not say anything about the almost sure convergence of posterior distributions for fixed samples of Y_n , but the necessary conditions in [29] are valid for also a.s. convergence. For example, the results in [29] imply that setwise convergence of the prior distributions is necessary for the setwise convergence of the posterior distributions (given samples of Y_n). We remark that samples of Y , not Y_n , are usually given. For the undominated case, a result of Berti et al [8] somewhat simplifies the study of posterior convergence. Under quite general conditions, their result reduces the problem of almost sure weak convergence of random measures to the study of only countably many sequences of conditional expectations. Piironen gave a sufficient condition that guarantees the convergence of posterior distributions when unknowns and observations are approximated [116]. The emphasis in his results was on obtaining with probability 1 the posterior convergence for the Souslin space-valued approximated unknowns given samples of multi-indexed observations of the corresponding approximated unknowns. His proof relies on improving the a.s. convergence of the conditional expectations of each function of the type $f(X_n)g(X)$, where f and g are continuous and bounded, to almost sure weak convergence of posterior distributions.

A concept close to the posterior convergence is the so-called discretization invariance, which was first used by Markku Lehtinen in 1990's (see [96]). It asks that the prior knowledge is consistent at all discretization levels and aims to the stability of posterior knowledge on different discretization levels. Definitions for discretization invariance in statistical inverse problems are given in [96, 97]. In [96], Lassas et al defined a proper linear discretization $X_n = P_n X$ of a Banach space-valued random variables X , where P_n are bounded linear operators on the Banach space F having finite-dimensional ranges and the random variables $\langle P_n X, \phi \rangle$ converge in distribution to $\langle X, \phi \rangle$ for all $\phi \in F'$. Gaussian priors and Besov space priors were shown to be discretization invariant in [96] in the sense that they have proper linear discretization for which the conditional mean estimates converge. An important example was studied by Lassas and Siltanen [97] who showed that the finite dimensional total variation priors converge to a Gaussian measure and the corresponding CM estimates converge to the CM estimate obtained with a Gaussian prior. The total variation priors are not discretization invariant as the finite-dimensional prior distributions lead to unwanted effects. A special method for obtaining stable posterior knowledge was suggested by Kaipio and Somersalo [76], who proposed the approximation error approach for statistical inverse problems. In approximation error approach, the conditioning random variable $Y = L(X) + \varepsilon$ is written as $Y = L(X_n) + (L(X) - L(X_n)) + \varepsilon$, where $L(X) - L(X_n)$ is taken to be an additional noise term $\tilde{\varepsilon}$. For example, if X is Gaussian and $X_n = P_n X$, where P_n are linear projection operators, the CM estimators take a consistent form $\mathbf{E}[X_n|Y] = P_n \mathbf{E}[X|Y]$. The problem becomes computationally more tractable if X_n and $\tilde{\varepsilon}$ are statistically independent in which case only the distribution of $\tilde{\varepsilon}$ needs to be additionally determined. This condition is often forced on $\tilde{\varepsilon}$ together with a numerically feasible approximated distribution [76, 140].

2. CONDITIONAL PROBABILITIES AND POSTERIOR DISTRIBUTIONS

2.1. Solution of the statistical inverse problem. We define what we exactly mean by a statistical inverse problem and its solution. We begin by recalling the definition of the conditional expectation.

The conditional expectation of $f \in L^1(\Omega, \Sigma, P)$ given a sub- σ -algebra $\Sigma_0 \subset \Sigma$ is a Σ_0 -measurable function $\mathbf{E}[f|\Sigma_0]$ such that

$$\int_A f dP = \int_A \mathbf{E}[f|\Sigma_0] dP$$

for all $A \in \Sigma_0$. Conditional expectations exist due to the Radon-Nikodym theorem as the densities of the (signed) measure $f dP$ with respect to the measure P on Σ_0 , but they are only defined up to sets $N \in \Sigma_0$ of P -measure zero. We denote $\mathbf{E}[\cdot|Y] = \mathbf{E}[\cdot|Y^{-1}(\mathcal{G})]$.

Definition 2.1. Let (Ω, Σ, P) be a complete probability space. Let F and G be two Souslin spaces equipped with their Borel σ -algebras \mathcal{F} and \mathcal{G} , respectively. Let $X : \Omega \rightarrow F$ and $Y : \Omega \rightarrow G$ be measurable mappings. We call a mapping $\mu : \mathcal{F} \times G \rightarrow [0, 1]$ a *solution of the statistical inverse problem of estimating the distribution of the unknown X given the observation Y* if

- (1) $\mu(U, Y(\omega)) = \mathbf{E}[1_U(X)|Y](\omega)$ P -almost surely for every $U \in \mathcal{F}$,
- (2) $y \mapsto \mu(U, y)$ is μ_Y -measurable for every $U \in \mathcal{F}$, and
- (3) $U \mapsto \mu(U, y)$ is a probability measure on (F, \mathcal{F}) for every $y \in G$.

The distributions $\mu(\cdot, y)$ are called *posterior distributions* of X given $Y = y$.

Strictly speaking, the posterior distributions are defined *a posteriori* of the observation $Y(\omega)$ but we feel that there is no harm in calling $\mu(\cdot, y)$ posterior distributions also for $y \notin R(Y)$ since $\mu_Y(G) = 1$.

The solution is just a regular conditional distribution of X given the sub- σ -algebra $Y^{-1}(\mathcal{G})$, where the regularity holds in the sense of Doob i.e. the solution μ is μ_Y -measurable in the second variable (see Remark 10.6.3 in [13] for a further discussion). The nature of the mapping $\omega \mapsto \mu(U, Y(\omega))$, which need not be $\sigma(Y)$ -measurable, is verified in the following simple lemma.

Lemma 2.2. Let (G, \mathcal{G}) be a measurable space. Let $Y : \Omega \rightarrow G$ be a measurable mapping from a complete probability space (Ω, Σ, P) into G . If $f : G \rightarrow \mathbf{R}$ is a μ_Y -measurable function then $f(Y)$ is a random variable on (Ω, Σ, P) , and $\mathbf{E}[f(Y)] = \int f(y) d\mu_Y(y)$. Moreover, if $\tilde{f} : G \rightarrow \mathbf{R}$ is a Borel measurable function such that $f = \tilde{f}$ μ_Y -a.s., then $f(Y(\omega)) = \tilde{f}(Y(\omega))$ P -almost surely and $\mathbf{E}[f(Y)|\Sigma_0](\omega) = \mathbf{E}[\tilde{f}(Y)|\Sigma_0](\omega)$ P -almost surely for any sub- σ -algebra $\Sigma_0 \subset \Sigma$.

Proof. Every μ_Y -measurable function has a Borel measurable version (see Proposition 2.1.11 in [13]). Denote with \tilde{f} a Borel measurable version of f . The set $N = \{y \in G : f(y) \neq \tilde{f}(y)\} \in \mathcal{G}^{\mu_Y}$ is then μ_Y -zero measurable and, by definition, there exists a μ_Y -zero measurable Borel set $B \in \mathcal{G}$ such that $N \subseteq B$. Especially, $Y^{-1}(N) \subset Y^{-1}(B)$, which has P -measure $P(Y^{-1}(B)) = \mu_Y(B) = 0$ so that also $Y^{-1}(N)$ belongs to the complete σ -algebra Σ and has P -measure zero. Therefore,

$$f(Y(\omega)) = \tilde{f}(Y(\omega))$$

P-almost surely. By the completeness of Σ , also the mapping $\omega \mapsto f(Y(\omega))$ is Σ -measurable. By the almost sure equivalence of the functions, we get

$$\mathbf{E}[f(Y(\omega))] = \mathbf{E}[\tilde{f}(Y(\omega))] = \int \tilde{f}(y) d\mu_Y(y) = \int f(y) d\mu_Y(y).$$

The conditional expectations of equivalent random variables coincide, since they have the same integrals over Σ_0 -measurable sets. \square

From the point of view of the posterior analysis, Condition 3 of Definition 2.1 may give a false sense of security. Any μ that satisfies Conditions 1 and 2 but is a probability measure only for μ_Y -a.e. y can be redefined on a negligible set in such a way that it is a solution. For example, if N is the μ_Y -zero measurable set that contains all y 's for which $\mu(\cdot, y)$ is not a probability measure, we may redefine $\mu(U, y)$ as $1_U(x_0)$ for some fixed $x_0 \in F$ and all $y \in N$. Then μ satisfies Conditions 1, 2 and 3, but $\mu(\cdot, y)$ is not related to the unknown when $y \in N$.

We briefly compare the solution μ with other formulations of Bayesian inverse problems. Clearly, any regular conditional distribution μ of X given Y (such that $y \mapsto \mu(U, y)$ is Borel-measurable for any $U \in \mathcal{F}$) qualifies as a solution. Especially, posterior distributions obtained by the Bayes formula (1) on \mathbf{R}^n for positive continuous probability densities form a solution of the form $\mu(U, y) = \int_U D(x|y) dx$ [75]. The Gaussian conditional probabilities in [94, 100, 102, 105] are also solutions that are allowed to be μ_Y -measurable in the sense of Condition 2. Our approach is similar to the work of Piiroinen [116], where a general formulation of the statistical inverse problem for Souslin space-valued random variables first appeared. The difference is that Piiroinen chose the posterior probabilities $\mu(U, y)$ to be universally measurable with respect to the second variable, that is, m -measurable for any finite Radon measure m on (G, \mathcal{G}) whereas we prefer to take all μ_Y -measurable versions as solutions, since it helps to avoid the somewhat artificial modifications of μ_Y -measurable functions (encountered for example in the Gaussian case [100]) to any universally measurable or \mathcal{G} -measurable functions. Lassas et al [96] used a different approach where the posterior distribution was obtained by defining reconstructors. A mapping $y \mapsto \mathcal{R}(g|y)$ is called a reconstructor of $g \in L^1(\mu_X)$ (more generally, a Bochner integrable g) given the observation Y if $\mathcal{R}(g, Y(\omega)) = \mathbf{E}[g(X)|Y^{-1}(\mathcal{G})](\omega)$ almost surely [96]. The concept of a reconstructor is more elemental than our solution. However, the reconstructors that were used for solving the statistical inverse problem in [63, 96] were chosen to be more regular. They depend continuously on observations and satisfy also Conditions 1 and 3. Hence, they form a regular conditional distribution and are especially solutions in the sense of Definition 2.1. A common point of the reconstructor and our solution is that both are defined for all $y \in G$, not only for samples $Y(\omega) \in R(Y)$. However, the simplicity of the reconstructors comes with some disadvantages. Namely, if the reconstructor of the unknown X does not originate from a regular conditional distribution, some power of the Bayesian inference is lost, as there is no posterior probability distribution to draw from. Furthermore, two reconstructors \mathcal{R}_1 and \mathcal{R}_2 of the same function f may differ on a "large" set $Y(N) \subset G$, where $N = \{\omega : \mathcal{R}_1(f, Y(\omega)) \neq \mathcal{R}_2(f, Y(\omega))\} \in \Sigma$ has probability zero. The set $Y(N)$ might not belong to \mathcal{G}^{μ_Y} and $Y(N)$ may have positive μ_Y -outer measure. Indeed, we provide a simple example of this situation with the help of the so-called image measure catastrophe (see p. 30 in [129]). Let $U \subset [0, 1]$ be a nonmeasurable set such that the Lebesgue outer measure $m^*(U) = 1$. Let $(U, \mathcal{B}_U([0, 1]), m|_U)$ be the restriction of the Lebesgue measure m on U i.e. the

Borel σ -algebra $\mathcal{B}_U([0, 1])$ contains all sets $B \cap U$, where $B \in \mathcal{B}([0, 1])$, and for such sets $m_U(B \cap U) = m(B \cap \tilde{U})$, where $\tilde{U} \in \mathcal{B}([0, 1])$ is such that $m^*(U) = m(\tilde{U})$, say $\tilde{U} = [0, 1]$. Take (Ω, Σ, P) to be the completion of $(U, \mathcal{B}_U([0, 1]), m|_U)$ and $F = G = [0, 1]$ equipped with its Borel σ -algebra. Let $Y : U \rightarrow [0, 1]$ be the identity and take $X = Y$. Then the image measure μ_Y is the Lebesgue measure on $[0, 1]$. Moreover, the conditional expectation of the measurable function $\omega \mapsto 1_{[0, 1]}(X(\omega))$ given $\sigma(Y)$ is

$$\mathbf{E}[1_{[0, 1]}(X)|\sigma(Y)](\omega) = 1_{[0, 1]}(Y(\omega))$$

which is equal to $1_U(Y(\omega))$ P -almost surely. However, the reconstructor $\mathcal{R}_1(1_{[0, 1]}, \cdot) : y \mapsto 1_U(y)$ is not μ_Y -measurable on $([0, 1], \mathcal{B}([0, 1]))$, and the two reconstructors $\mathcal{R}_1(1_{[0, 1]}, \cdot)$ and $\mathcal{R}_2(1_{[0, 1]}, \cdot) : y \mapsto 1_{[0, 1]}(y)$ differ on the set $N_0 := [0, 1] \setminus U$ which has positive μ_Y -outer measure. Condition 2 helps us to avoid this small shortcoming. If the reconstructors are μ_Y -measurable, the set $N_0 = \{y \in G : \mathcal{R}_1(f, \cdot) \neq \mathcal{R}_2(f, \cdot)\} \in \mathcal{G}^{\mu_Y}$ and $\{\omega : \mathcal{R}_1(f, Y(\omega)) \neq \mathcal{R}_2(f, Y(\omega))\} = Y^{-1}(N_0)$. Then N_0 has zero μ_Y -measure.

A regular conditional distribution is not unique in general because of the non-uniqueness of the conditional expectations. For our theoretical considerations, the following concept (adapted from [13] in context of regular conditional measures) is useful.

Definition 2.3. We say that a solution μ of the statistical inverse problem of estimating the distribution of X given the observation Y is *essentially unique* if for any other solution $\tilde{\mu}$ of the same statistical inverse problem there exists a set $C = C(\mu, \tilde{\mu}) \in \mathcal{G}^{\mu_Y}$ with $\mu_Y(C) = 1$ such that $\tilde{\mu}$ agrees with μ on $\mathcal{F} \times C$. Similarly, we say that the posterior distribution $\mu(\cdot, y)$ is essentially unique if μ is essentially unique.

In other words, an essentially unique solution μ may be arbitrary on the sets of the form $\mathcal{F} \times N$, where $N \subset G$ is a set of μ_Y -measure zero. In a sense, this makes the posterior distribution $\mu(\cdot, Y(\omega))$ a relevant estimate of the distribution of X with probability 1.

Next, we recall some results on the existence and essential uniqueness of regular conditional distributions in Souslin spaces. The existence of regular conditional distributions of X given Y has been shown in Lemma 4.2 of [116] (by using the definition of the Souslin space and the existence of regular conditional distributions on Polish spaces, leading to a universally measurable kernel μ), and in Example 10.7.5 of [13], where also the essential uniqueness has been verified. The present "extension" covers μ_Y -measurable solutions. The condensed proof is included only to support the last sentence, which provides some motivation for the main results of this work. Namely, the definition of the conditional expectation may give the impression that we need to specify some random variable Y among all equivalent random variables for determining the conditional expectation of $1_U(X)$ when an observation $y = Y(\omega_0) \in G$ has occurred. This is not true as a weaker description of Y and X suffices.

Theorem 2.4. *Let (F, \mathcal{F}) and (G, \mathcal{G}) be two measurable spaces. Let X be an F -valued random variable and Y be a G -valued random variable on a complete probability space (Ω, Σ, P) . If F and G are Souslin spaces equipped with their Borel σ -algebras, then there exists an essentially unique solution $\mu : \mathcal{F} \times G \rightarrow [0, 1]$ of the statistical inverse problem of estimating the distribution of the unknown X given the observation Y .*

The values $\mu(U, y)$ are determined by the joint distribution $\mu_{(X,Y)}$ of X and Y for all $U \in \mathcal{F}$ and μ_Y -almost every $y \in G$.

Proof. First we show that for each $U \in \mathcal{F}$ there exists a solution $\mu_0(U, \cdot) : G \rightarrow [0, 1]$ such that $y \mapsto \mu_0(U, y)$ is Borel-measurable and $\omega \mapsto \mu_0(U, Y(\omega))$ is a conditional expectation of $1_U(X)$ given $Y^{-1}(\mathcal{G})$.

Consider the measure space $(F \times G, \mathcal{B}(F \times G), \mu_{(X,Y)})$ and the sub- σ -algebra $\mathcal{G}_0 = \{\emptyset, F\} \otimes \mathcal{G}$ generated by the canonical projection $p_2(x, y) = y$ to the second variable. Recall, that the direct products of Souslin spaces are Souslin spaces. Due to the Souslin property of $F \times G$, there exists a conditional measure $\mu_0 : \mathcal{B}(F \times G) \times (F \times G) \rightarrow [0, 1]$ such that $\mu_0(U', \cdot)$ is \mathcal{G}_0 -measurable for every $U' \in \mathcal{B}(F \times G)$, the measure $\mu(\cdot, (x, y))$ is a probability distribution on $\mathcal{B}(F \times G)$ for every $(x, y) \in G$, and

$$\mu_{(X,Y)}(U' \cap V') = \int_{V'} \mu_0(U', (x, y)) d\mu_{(X,Y)}(x, y)$$

for every $U' \in \mathcal{B}(F \times G)$ and $V' \in \mathcal{G}_0$ by Corollary 10.4.6 in [13]. Let us restrict $\mu_0(U', (x, y))$ on sets U' of the form $U \times G$, where $U \in \mathcal{F}$. Since \mathcal{G}_0 is trivial with respect to the first variable, we may denote the restriction with $\mu_0(U, y)$ where $U \in \mathcal{F}$ and $y \in G$. Especially, $y \mapsto \mu_0(U, y)$ is \mathcal{G} -measurable and

$$P(X \in U \cap Y \in V) = \mu_{(X,Y)}(U \times V) = \int_V \mu_0(U, y) d\mu_Y(y) = \int_{Y^{-1}(V)} \mu_0(U, Y) dP$$

for every $U \in \mathcal{F}$ and $V \in \mathcal{G}$. Therefore, $\mu_0 : \mathcal{F} \times G \rightarrow [0, 1]$ is a solution of the statistical inverse problem of estimating the probabilities of the unknown X given the observation Y .

A solution μ is essentially unique since the Borel σ -algebra of a Souslin space is countably generated (see [13]). Indeed, suppose that μ and ν are two solutions in the sense of Definition 2.1. For $U \in \mathcal{F}$, we have that $\mu(U, Y(\omega)) = \mathbf{E}[1_U(X) | Y^{-1}(\mathcal{G})](\omega) = \nu(U, Y(\omega))$ P -almost surely. Then $\mu(\cdot, y) = \nu(\cdot, y)$ outside some μ_Y -zero measurable set $N_U \in \mathcal{G}^{\mu_Y}$, since

$$0 = \mathbf{E}[|\tilde{\mu}(U, Y) - \tilde{\nu}(U, Y)|] = \int |\mu(U, y) - \nu(U, y)| d\mu_Y(y)$$

by Lemma 2.2. Every countable algebra \mathcal{F}_0 that generates the σ -algebra \mathcal{F} is measure-determining, i.e. measures coinciding on \mathcal{F}_0 coincide on \mathcal{F} (e.g. Lemma 1.9.4 in [13]). Hence, the two solutions coincide except for $y \in \cup_{U \in \mathcal{F}_0} N_U$.

Finally, if μ is any solution then the values $\mu(U, y)$ are determined by the measure $\mu_{(X,Y)}$ for all $U \in \mathcal{F}$ and μ_Y -almost all $y \in G$ since μ coincides with μ_0 on $\mathcal{F} \times C$ (by essential uniqueness) and the values of $\mu_0(U, \cdot)$ are actually versions of the Radon-Nikodym densities of measures $\mu_{(X,Y)}(U, \cdot)$ with respect to μ_Y for μ_Y -almost all y . The distribution μ_Y is the marginal of $\mu_{(X,Y)}$. \square

We have reached the usual starting point of nonparametric Bayesian statistics. In a conventional Bayesian experiment, one specifies only conditional distributions of Y given $X = x$ for all values $x \in G$ – the so-called parametric family of distributions or sampling distributions – and the prior distribution μ_X on (F, \mathcal{F}) [45, 128], which together determine the joint distribution of X and Y .

Remark 1. The choice of the sample space (G, \mathcal{G}) of random variable Y is usually not trivial. One might choose as well a larger (or sometimes even a smaller) space than G . The solutions of the statistical inverse problem could, in principle, depend on the choice of the sample space (G, \mathcal{G}) since the conditioning σ -algebra $Y^{-1}(\mathcal{G})$

depends on the topology of G . But since we are working with the Souslin spaces this is not the case. Indeed, if (G_1, \mathcal{G}_1) and (G_2, \mathcal{G}_2) are two Souslin spaces equipped with their Borel σ -algebras and $i : G_1 \mapsto G_2$ is a continuous (or just Borel!) injection, then, quite remarkably, $i^{-1}(\mathcal{G}_2) = \mathcal{G}_1$. Indeed, $i^{-1}(\mathcal{G}_2) \subset \mathcal{G}_1$ by the continuity of i . Moreover, the image of a Borel set under a Borel mapping between Souslin spaces is a Souslin set i.e. a Souslin space with respect to the relative topology (see Theorem 6.7.3 in [13]). Therefore, $i(G_1)$, $i(B)$ and $i(G_1 \setminus B)$ are all Souslin sets in G_2 for any Borel set $B \in \mathcal{G}_1$. By injectivity, $i(G_1 \setminus B) = i(G_1) \setminus i(B)$ i.e. the complement of the Souslin set $i(B)$ in the subspace $i(G_1)$ of G_2 is a Souslin set. By Corollary 6.6.10 in [13], a Souslin set in a Hausdorff space is a Borel set if also its complement is a Souslin set. Therefore, $i(B)$ is a Borel set in the relative topology of $i(G_1)$. But each Borel set $i(B)$ in $i(G_1)$ is of the form $i(B) = i(G_1) \cap B'$, where $B' \in \mathcal{G}_2$. Therefore, $B = i^{-1}(B')$ for some $B' \in \mathcal{G}_2$ which implies that $\mathcal{G}_1 = i^{-1}(\mathcal{G}_2)$. Consequently, $(iY)^{-1}(\mathcal{G}_2) = Y^{-1}(\mathcal{G}_1)$ for any G_1 -valued random variable Y . Therefore, $\mu_1(\cdot, Y(\omega)) = \mu_2(\cdot, i(Y(\omega)))$ P -almost surely for any solutions μ_1 and μ_2 of the inverse problems of estimating the distribution of X given $Y : \Omega \rightarrow G_1$ and $i(Y)$, respectively. If $\tilde{\mu}_2$ is a Borel measurable version of μ_2 , then $\tilde{\mu}_2(U, i(y))$ is \mathcal{G}_1 -measurable and $\mu_2(U, y') = \tilde{\mu}_2(U, y')$, except possibly on some set N such that $\mu_{i(Y)}(N) = 0$ which implies that $\mu_2(U, i(y)) = \tilde{\mu}_2(U, i(y))$ except possibly on the set $i^{-1}(N)$ which has $\mu_Y(i^{-1}(N)) = 0$. Therefore, $\mu_2(\cdot, i(\cdot))$ is also a solution of the statistical inverse problem of estimating the distribution of X given Y . We are allowed to mis-specify the Souslin sample space G_1 by Borel injections without altering the essentially unique solution. In the general case that involves non-Souslin spaces, we only know that $i^{-1}(\mathcal{G}_2) \subset \mathcal{G}_1$, where the inclusion may be strict. As an example, take G_1 and G_2 to be the sequence space ℓ^∞ where we take \mathcal{G}_1 to be the usual Borel σ -algebra with respect to the supremum norm topology (which is not separable) and \mathcal{G}_2 to be the Borel σ -algebra with respect to the weak topology, and take i to be the identity. Then $i^{-1}(\mathcal{G}_2) \neq \mathcal{G}_1$ (see Proposition 2.9 in [147]).

2.2. Partial uniqueness of the solution. From practical point of view, the essential uniqueness is not enough since we are given some fixed observation $y_0 \in G$ that might belong to the set where arbitrariness of μ still rules. Our proposal for removing this deficiency of the posterior distributions is to proceed as in the finite-dimensional case, where μ is required to depend continuously on the second variable i.e. the posterior distributions depend continuously on the observations. The following new concept turns out to be useful.

Definition 2.5. Let (Ω, Σ, P) be a complete probability space. Let F and G be two Souslin spaces equipped with their Borel σ -algebras \mathcal{F} and \mathcal{G} , respectively. Let $X : \Omega \rightarrow F$ and $Y : \Omega \rightarrow G$ be measurable mappings. Let μ be a solution of the statistical inverse problem of estimating the distribution of the unknown X given the observation Y . Let $A \subset G$ and let $\mathcal{F}_0 \subset \mathcal{F}$. We say that a solution μ is \mathcal{F}_0 -continuous on A if the mapping $y \mapsto \mu(U, y)$ is continuous on A with respect to the relative topology for every $U \in \mathcal{F}_0$.

Consider a set $S \subset G$ that contains every point $y \in G$ whose any open neighborhood has positive μ_Y -measure. On Souslin spaces such a set S is known to coincide with the *topological support* of μ_Y , i.e. the smallest closed set $S \subset G$ such that $\mu_Y(S) = 1$

Lemma 2.6. *Let ν be a Borel probability measure on a Souslin space G . The topological support of ν exists and it consists of exactly those $y \in G$ whose every open neighborhood has positive measure.*

Proof. See Theorem 2.1 in [113], which generalizes to Souslin spaces, since Souslin spaces are hereditarily Lindelöf by Lemma 6.6.4 of [13]. \square

We obtain partial uniqueness of the posterior distributions by using the continuity of solutions on certain subsets of the topological support of μ_Y . We denote with A° the interior points of A .

Theorem 2.7. *Let F and G be Souslin spaces equipped with their Borel σ -algebras \mathcal{F} and \mathcal{G} respectively. Let X be an F -valued and Y be a G -valued random variable on a complete probability space (Ω, Σ, P) . Let $A \in \mathcal{G}^{\mu_Y}$ be a subset of the topological support S of μ_Y such that either $A \subset \overline{A^\circ}$ or $\mu_Y(A) = 1$. Let $\mathcal{F}_0 \subset \mathcal{F}$ be a measure-determining class.*

All solutions of the statistical inverse problem of estimating the probabilities of X given Y that are \mathcal{F}_0 -continuous on A coincide on $\mathcal{F} \times A$.

Proof. Assume that μ_1 and μ_2 are two solutions that have the described properties. If $\mu_1 \neq \mu_2$ on $\mathcal{F} \times A$ then there exists $y_0 \in A$ and $U_0 \in \mathcal{F}$ such that $\mu_1(U_0, y_0) \neq \mu_2(U_0, y_0)$, say $\mu_1(U_0, y_0) - \mu_2(U_0, y_0) > \varepsilon$. Since $\mu_i(\cdot, y_0)$, $i = 1, 2$, are measures, the set U_0 can be taken to be from the measure-determining class \mathcal{F}_0 .

The function $f : A \rightarrow \mathbf{R}$ defined as $f(y) := \mu_1(U_0, y) - \mu_2(U_0, y)$ is continuous in the relative topology of A and positive at y_0 . The set $f^{-1}((\varepsilon, \infty))$ is therefore a non-empty open neighborhood of y_0 in the relative topology of A , and there exists a non-empty open set $V \subset G$ such that $V \cap A = f^{-1}((\varepsilon, \infty))$. The set $V \cap A$ has positive μ_Y -measure. Indeed, if $\mu_Y(A) = 1$, then $\mu_Y(V \cap A) = \mu_Y(V) > 0$ by Lemma 2.6, since $y_0 \in V \cap A$ belongs also to the support of μ_Y . On the other hand, if $A \subset \overline{A^\circ}$, the neighborhood V of y_0 contains also points from A° . It follows that $V \cap A$ contains a non-empty open set $V \cap A^\circ$. By Lemma 2.6, $\mu_Y(V \cap A) > 0$. This implies that $\mu_1(U_0, y) - \mu_2(U_0, y) > \varepsilon$ on a set $f^{-1}((\varepsilon, \infty)) \in \mathcal{G}^{\mu_Y}$ of positive μ_Y -measure. Therefore, it is impossible that the both mappings μ_1 and μ_2 satisfy the requirements of Definition 2.1, in particular the property

$$\int_{f^{-1}((\varepsilon, \infty))} \mu_i(U_0, y) d\mu_Y(y) = P(X \in U_0 \cap Y \in f^{-1}((\varepsilon, \infty))), \quad i = 1, 2,$$

of conditional expectations does not hold. Hence, the two solutions necessarily coincide on $\mathcal{F} \times A$. \square

Remark 2. Recall, that a Borel measure is called strictly positive if it is positive on all non-empty open subsets. Then the topological support of the measure is the whole space. When μ_Y is strictly positive, the partial uniqueness holds on $\mathcal{F} \times G$ for the solutions that are \mathcal{F}_0 -continuous on G . If μ_Y is strictly positive and the solution μ is \mathcal{F}_0 continuous on some non-empty open subset A of G , we get similarly the uniqueness on $\mathcal{F} \times A$. This situation is often encountered in finite-dimensional statistical inverse problems, where one usually excludes those $y \in G$ for which the continuous probability density function of Y vanishes.

Remark 3. The partial uniqueness of the solution is obtained by fixing the topology of the space of observations. However, the topology of a Souslin space is a slightly ambiguous concept in measure theoretical sense. Namely, it is well-known that

different topologies can generate the same Borel sets. For example, any Borel measurable function on a Souslin space is continuous with respect to some stronger topology that makes the space Souslin and generates the same Borel sets as the original topology (see Exercise 6.10.62 in [13] for the proof). If μ is a \mathcal{F} -continuous on a Souslin space and $\tilde{\mu}$ is its Borel-measurable version that is not continuous, then the both are continuous with respect to some stronger topology that makes also $\tilde{\mu}$ continuous. We remark that although the essentially unique solutions are invariant under injective continuous mappings between Souslin spaces (see Remark 1), the strengthening of the topology can affect the partial uniqueness of the solution e.g. by diminishing the topological support.

Due to the properties of the conditional expectation, the prior distribution $\mu_X(U)$ is the mixture $\int \mu(U, y) d\mu_Y(y)$ of all posterior distributions so that the prior probability of U vanishes exactly when μ_Y -almost all posterior probabilities of U vanish. When μ is regular enough, we get the following converse result, which contrasts nicely with the well-known representation theorem considered in the next section.

Theorem 2.8. *Let F and G be Souslin spaces equipped with their Borel σ -algebras \mathcal{F} and \mathcal{G} respectively. Let X be an F -valued and Y be a G -valued random variable on a complete probability space (Ω, Σ, P) . Let $A \in \mathcal{G}^{\mu_Y}$ be any subset of the topological support S of μ_Y such that either $A \subset \overline{A}^\circ$ or $\mu_Y(A) = 1$. If μ is a solution of the statistical inverse problem of estimating probabilities of X given Y that is \mathcal{F} -continuous on A , then the posterior distribution $\mu(\cdot, y)$ at any $y \in A$ is absolutely continuous with respect to the prior distribution.*

Proof. Assume that $\mu_X(U) = 0$ for some $U \subset \mathcal{F}$. According to the definition of conditional expectation,

$$(6) \quad \int \mu(U, y) d\mu_Y(y) = P(X \in U) = \mu_X(U),$$

which now vanishes. Since the solution is non-negative, we get that $\mu(U, y) = 0$ μ_Y -almost surely on G . Since $y \mapsto \mu(U, y)$ is continuous on A , the set $\tilde{V} = \{y \in A : \mu(U, y) > 0\}$ is a relatively open set i.e. there exist an open set $V \subset G$ such that $V \cap A = \tilde{V}$. Suppose \tilde{V} is non-empty. Similarly as in the proof of Theorem 2.7, $\mu_Y(\tilde{V}) = \mu_Y(V) > 0$ when A has full measure and $\mu_Y(\tilde{V}) > 0$ when $A \subset \overline{A}^\circ$. But this contradicts (6) because $\mu_X(U) = 0$. Thus the set \tilde{V} is empty and $\mu(U, y) = 0$ for all y from A . \square

When Theorem 2.8 holds for $A = S$, every Borel set B with full prior probability has also full posterior probability $\mu(B, y) = 1$ for all $y \in S$. Our posterior perception of the unknown appears to be inline with our prior insight in this aspect.

Remark 4. According to a result of Macci [103], the absolute continuity of the posterior distributions $\mu(\cdot, y)$ with respect to the prior distribution for μ_Y -almost every y implies that the conditional distribution of Y given $X = x$ is absolutely continuous with respect to ($=$ dominated by) μ_Y for μ_X -a.e. $x \in F$. By Theorem 2.8, the posterior probabilities of Borel sets may depend continuously on the observations $y \in G$ only in the dominated cases i.e the conditional distribution of Y given $X = x$ has to be absolutely continuous with respect to some σ -finite measure for μ_X -a.e. $x \in F$. The same conclusion holds even if the space G is replaced with some subset $A \in \mathcal{G}^{\mu_Y}$ having full μ_Y -measure. In the undominated cases, the

posterior distribution has necessarily a large amount of discontinuities – the set of all discontinuity points must have positive μ_Y -measure.

A partial converse to Theorem 2.8 holds in complete separable metric spaces.

Theorem 2.9. *Let F and G be complete separable metric spaces equipped with their Borel σ -algebras \mathcal{F} and \mathcal{G} , respectively. Let X be an F -valued and Y be a G random variable on a complete probability space (Ω, Σ, P) . Let μ be a solution of the statistical inverse problem of estimating the distribution of X given Y . If the family of the posterior distributions $\{\mu(\cdot, y) : y \in G\}$ is dominated by a Borel measure ν on \mathcal{F} , then for every $\epsilon > 0$ there exists a compact set $K = K(\epsilon, \mu) \in \mathcal{G}$ such that $\mu_Y(G \setminus K) < \epsilon$ and μ is \mathcal{F}_0 -continuous on K for some family \mathcal{F}_0 of measure-determining sets.*

Proof. Equip the space M of all probability measures on (F, \mathcal{F}) with the topology of the weak convergence (i.e. convergence of integrals of all bounded continuous functions). It is well-known that this space is a complete separable metric space whenever F is a complete separable metric space (see Theorem 8.9.5 in [13]). Equip M with the Borel σ -algebra \mathcal{M} with respect to the weak topology i.e. the cylinder set σ -algebra of the sets of the type

$$\{\nu \in \mathcal{M} : (\nu(f_1), \nu(f_2), \dots) \in B\},$$

where $f_i, i \in \mathbf{N}$ are continuous bounded functions on F and $B \in \mathcal{B}(\mathbf{R}^\infty)$ (with respect to the coordinate-wise convergence). By Condition 2 of Definition 2.1, the solutions $y \mapsto \mu(\cdot, y)$ are μ_Y -measurable mappings from G to M since $y \mapsto \mu(f, y)$ is μ_Y -measurable as a pointwise limit of integrals of simple functions. By the Lusin theorem (see Theorem 7.1.13 in [12]), there exists a family of compact sets $\mathcal{K} \subset \mathcal{G}$ such that given any $\epsilon > 0$, the probability $\mu_Y(K^C) < \epsilon$ for some $K \in \mathcal{K}$ and the measure-valued random variable $y \mapsto \mu(\cdot, y)$ is continuous on K in the weak topology, of measures implying that $\lim_{i \rightarrow \infty} \mu(f, y_i) = \mu(f, y)$ whenever $\lim_{i \rightarrow \infty} y_i = y$ in K . Especially, the mappings $y \mapsto \mu(\cdot, y)$ are \mathcal{F}_0 -continuous on K , where \mathcal{F}_0 consists of all Borel sets U whose boundary satisfies $\mu(\partial U, y) = 0$ for all $y \in K$. This follows from the fact that $\lim_{i \rightarrow \infty} \mu(U, y_i) = \mu(U, y)$ whenever $\lim_{i \rightarrow \infty} y_i = y$ by the weak convergence (see Corollary 8.2.10 in [13]). If the family of posterior distributions $\{\mu(\cdot, y) : y \in K\}$ is dominated by some Borel measure, then \mathcal{F}_0 is a measure-determining set (see Lemma 1.9.4 and Proposition 8.2.8 in [13]). \square

3. THE REPRESENTATION OF POSTERIOR DISTRIBUTIONS

In this section, we consider a known representation formula (see Section 1.2.2 in [45], Theorem 1.31 in [128], or pp. 231–232 in [132]) for solution of the statistical inverse problem of estimating the distribution of X given the observations Y that generalizes the finite-dimensional formula

$$D(x|y) = CD(y|x)D_{pr}(x).$$

For readers convenience, the proofs of Lemma 3.1, Lemma 3.2 and Theorem 3.3 are included, although they are special cases of more general known results.

Throughout the section, we assume that F and G are locally convex Souslin topological vector spaces equipped with their Borel σ -algebras \mathcal{F} and \mathcal{G} , respectively, and X is taken to be an F -valued random variable and ε is taken to be a G -valued random variable statistically independent from X . All the random variables are

defined on the same complete probability space (Ω, Σ, P) . The mapping $L : F \rightarrow G$ is assumed to be continuous. We denote $Y = L(X) + \varepsilon$.

First, we check that Y is indeed a random variable as a combination of Borel measurable mappings. The product space $F \times G$ is equipped with the usual product σ -algebra $\mathcal{F} \otimes \mathcal{G}$ generated by rectangles $U \times V$, where $U \in \mathcal{F}$ and $V \in \mathcal{G}$.

Lemma 3.1. *The mapping $T : (x, z) \mapsto L(x) + z$ is Borel measurable from $F \times G$ to G .*

Proof. As the addition is just $\mathcal{B}(G \times G)$ -measurable by continuity, there is the question whether the Borel σ -algebra $\mathcal{B}(G \times G)$ of the topological product space coincides with the product σ -algebra $\mathcal{G} \otimes \mathcal{G}$ generated by the rectangles $V \times W$ where $V, W \in \mathcal{G}$.

Certainly, $\mathcal{G} \otimes \mathcal{G} \subset \mathcal{B}(G \times G)$, since the products of open sets $V, W \subset G$ form a basis of topology for $G \times G$.

Due to the Souslin property, the space $G \times G$ is hereditarily Lindelöf ([13], Lemma 6.6.4 and Lemma 6.6.5). Any open set in $G \times G$ can therefore be expressed as a countable union of sets of the form $V \times W$, where $V, W \in \mathcal{G}$ are open. Hence $\mathcal{B}(G \times G) \subset \mathcal{G} \otimes \mathcal{G}$. \square

We verify now that for any μ_Y -integrable $f : G \rightarrow \mathbf{R}$, the conditional expectation of $f(Y)$ given X is the random variable

$$\mathbf{E}[f(Y)|X](\omega) = \int_G f(z) d\mu_{\varepsilon+L(X(\omega))}(z).$$

Here the measure $\mu_{\varepsilon+L(X(\omega))}$ is the image measure of the random variable $\omega' \mapsto \varepsilon(\omega') + L(X(\omega))$, where $X(\omega)$ is treated as a constant. We apply the following more general claim, for which we failed to find a reference.

Lemma 3.2. *Let Z_1 be an F -valued and Z_2 be a G -valued random variable that are statistically independent. Denote $Z_3 = T(Z_1, Z_2)$, where $T : F \times G \rightarrow G$ is a Borel measurable mapping. For any μ_{Z_3} -integrable function $f : G \rightarrow \mathbf{R}$, it holds that*

$$\mathbf{E}[f(Z_3)|Z_1](\omega) = \int_G f(z) d\mu_{T(Z_1(\omega), Z_2)}(z)$$

P -almost surely, and $\int_G f(z) d\mu_{T(Z_1(\omega), Z_2)}(z)$ is a version of the conditional expectation of $f(Z_3)$ given $\sigma(Z_1)$.

Proof. We show that the claim holds for a Borel measurable version of f , which exists by Proposition 2.1.11 in [13]. The generalization for μ_{Z_3} -measurable functions follows then from Lemma 2.2.

Remark that $f \circ T : F \times G \rightarrow \mathbf{R}$ is then a Borel measurable function. We will show that $\mathbf{E}[g(Z_1, Z_2)|Z_2](\omega) = \int_G g(Z_1(\omega), z_2) d\mu_{Z_2}(z_2)$ holds for all Borel measurable simple functions g on $F \times G$. The usual approximation of Borel measurable functions with simple functions implies then for $g = f \circ T$ that

$$\begin{aligned} \mathbf{E}[f(T(Z_1, Z_2))|Z_1](\omega) &= \int_G f(T(Z_1(\omega), z_2)) d\mu_{Z_2}(z_2) = \mathbf{E}[f(T(Z_1(\omega), Z_2))] \\ &= \int_G f(z) d\mu_{T(Z_1(\omega), Z_2)}(z). \end{aligned}$$

Take now $g = 1_C$, where $C \in \mathcal{B}(F \times G)$. We need to determine the conditional expectation $\mathbf{E}[1_C(Z_1, Z_2)|Z_2]$ i.e. the conditional distribution of (Z_1, Z_2) given $\sigma(Z_2)$.

Since F and G are Souslin spaces, a regular conditional measure exists (by Corollary 10.4.6 in [13]) and is determined by values on any measure-determining sets. In Souslin spaces, the rectangular sets $C = B_1 \times B_2$, where $B_1 \in \mathcal{F}$ and $B_2 \in \mathcal{G}$ are measure-determining sets, since $\mathcal{B}(F \times G) = \mathcal{F} \otimes \mathcal{G}$ (see the proof of Lemma 3.1 and Lemma 1.9.4 in [13]). By the properties of the conditional expectation,

$$\begin{aligned} \mathbf{E}[1_{B_1 \times B_2}(Z_1, Z_2)|Z_1](\omega) &= 1_{B_1}(Z_1(\omega)) \int 1_{B_2}(z_2) d\mu_{Z_2}(z_2) \\ &= \int 1_{B_1 \times B_2}(Z_1(\omega), z_2) d\mu_{Z_2}(z_2). \end{aligned}$$

□

Here is the description of the solutions $\mu(U, z)$ modulo μ_Y -zero measurable sets. The result is a special case of Kallianpur-Striebel formula [79].

Theorem 3.3. *Let $\mu_{\varepsilon+L(x)}$ be absolutely continuous with respect to a σ -finite measure ν for μ_X -a.e. $x \in F$. Set*

$$\rho(x, z) = \begin{cases} \frac{d\mu_{\varepsilon+L(x)}}{d\nu}(z) & \text{when } \mu_{\varepsilon+L(x)} \ll \nu \\ 0 & \text{otherwise.} \end{cases}$$

If $\rho(x, z)$ is a non-negative $\mu_X \times \nu$ -measurable function on $F \times G$, then there is an essentially unique solution μ of the statistical inverse problem of estimating the distribution of X given $Y = L(X) + \varepsilon$ such that

$$(7) \quad \mu(U, z) = \frac{\int 1_U(x) \rho(x, z) d\mu_X(x)}{\int \rho(x, z) d\mu_X(x)}$$

for all $z \in G \setminus N_0$, where the set

$$N_0 = \{z \in G : \int \rho(x, z) d\mu_X(x) = 0 \text{ or } \infty\}$$

has μ_Y -measure zero.

If $\mu_Y(N) = 0$ then N is also $\mu_{\varepsilon+L(x_0)}$ -zero measurable for μ_X -almost every $x_0 \in F$. If additionally $\mu_{\varepsilon+L(x)} \ll \nu$ for all $x \in F$ and ρ is positive $\mu_X \times \nu$ -almost everywhere, then $\mu_{\varepsilon+L(x_0)}(N) = 0$ for all $x_0 \in F$.

Proof. Let $\mu(U, z)$ be defined by (7). If $z \in N_0$, we set $\mu(U, z) = 1_U(x_0)$ for some fixed $x_0 \in F$. We prove that μ is a solution.

Let $U \in \mathcal{F}$ and $V \in \mathcal{G}$. By Theorem 2.4 there exists an essentially unique solution, which we denote here with $\tilde{\mu}$. We write two expressions for $P(X \in U \cap Y \in V)$ using Lemma 3.2. The first is

$$\begin{aligned} \mathbf{E}[1_U(X) \mathbf{E}[1_V(Y)|X]] &= \int 1_U(x) \left(\int 1_V(z) d\mu_{\varepsilon+L(x)}(z) \right) d\mu_X(x) \\ (8) \quad &= \int 1_U(x) \left(\int 1_V(z) \rho(x, z) d\nu(z) \right) d\mu_X(x) \\ &= \int 1_V(z) \left(\int 1_U(x) \rho(x, z) d\mu_X(x) \right) d\nu(z) \end{aligned}$$

and the second expression is

$$\begin{aligned}
(9) \quad & \mathbf{E}[1_V(Y)\mathbf{E}[1_U(X)|Y]] = \mathbf{E}[1_V(Y)\tilde{\mu}(U, Y)] \\
& = \mathbf{E}[\mathbf{E}[1_V(Y)\tilde{\mu}(U, Y)|X]] \\
& = \int \int 1_V(z)\rho(x, z)\tilde{\mu}(U, z)d\nu(z)d\mu_X(x) \\
& = \int 1_V(z) \left(\int \rho(x, z)d\mu_X(x) \right) \tilde{\mu}(U, z)d\nu(z).
\end{aligned}$$

The measurability of ρ is used in changing the order of integrations by the Fubini theorem. The integrability of ρ follows automatically from the finiteness of the left-hand side of (9) for $U = F$ and $V = G$.

Since the equivalence of (8) and (9) holds for all $V \in \mathcal{G}$, we obtain

$$\tilde{\mu}(U, z) \int \rho(x, z)d\mu_X(x) = \int 1_U(x)\rho(x, z)d\mu_X(x)$$

for ν -almost every z . Hence, $\tilde{\mu}(U, z) = \mu(U, z)$ for ν -almost every y such that $0 < \int \rho(x, z)d\mu_X(x) < \infty$.

The denominator in (7) may vanish only on a set A of μ_Y -measure zero since the choice $U = F$, $V = A$ gives $\mu_Y(A) = 0$ in (8). The same consideration implies that also the measure μ_Y is absolutely continuous with respect to ν . Similarly, the denominator is finite ν -almost surely, which implies μ_Y -almost surely. We conclude that N_0 has μ_Y -measure zero and $\tilde{\mu}(U, z) = \mu(U, z)$ μ_Y -almost surely. Then $\mu(U, y)$ satisfies Condition 2 of Definition 2.1. By Lemma 2.2, μ satisfies Condition 1. By the integrability of ρ , μ satisfies Condition 3 of Definition 2.1.

We proceed to the last claim. Taking $V = N$ and $U \in \mathcal{F}$ in (8) implies that $\mu_{\varepsilon+L(x)}(N) = \int 1_N(z)\rho(x, z)d\nu(z)$ vanishes for μ_X -almost all $x \in F$. When ρ is a.e. positive, also $\nu(N)$ has to vanish. We obtain $\mu_{L(x_0)+\varepsilon}(N) = 0$ for all $x_0 \in F$ by using the absolute continuity. \square

The last statement of the above theorem is added to show how small the zero measurable set for a given unknown is. The representation formula does not improve the essential uniqueness of solutions, because the Radon-Nikodym density $z \mapsto d\mu_{L(x)+\varepsilon}/d\nu(z)$ is only determined up to ν -equivalence. It should be noted that under the domination assumptions on $\mu_{\varepsilon+L(x)}$ in Theorem 3.3, there always exists versions of the Radon-Nikodym densities that are jointly measurable. In [79], this claim is proved assuming that $Y^{-1}(\mathcal{G})$ is countably generated. In Souslin spaces, the Borel σ -algebras are countably generated by Corollary 6.7.5 in [13].

It is easy to see, that the prior distribution μ_X and the posterior distribution $\mu(\cdot, z)$ are equivalent if (7) holds and $\rho(\cdot, z) > 0$ μ_X -almost everywhere.

Remark 5. The existence of ν is a delicate matter. For example, the measure $\mu_{\varepsilon+L(x)}$ may not be almost surely absolutely continuous with respect to μ_Y , although

$$\mu_Y(U) = \mathbf{E}[1_U(Y)] = \mathbf{E}[\mathbf{E}[1_U(Y)|X]] = \int \mu_{\varepsilon+L(x)}(U)d\mu_X(x)$$

by Lemma 3.2. We can only conclude that $\mu_{\varepsilon+L(x)}(U)$ vanishes μ_X -a.s. whenever $\mu_Y(U)$ vanishes and the μ_X -zero measurable set may depend on U . A Gaussian example in Remark 10 of Section 5 shows that this is indeed the case. In general, the Halmos-Savage theorem (Lemma 7 in [59]), states that from a dominated family

of finite measures, which in our case is $\{\mu_{\varepsilon+L(x)} : x \in M\}$ where $\mu_X(M) = 1$, one can pick out countably many measures $\mu_{\varepsilon+L(x_i)}$ in such a way that the measure $\nu := \sum_i a_i \mu_{\varepsilon+L(x_i)}$, where $\sum_i a_i = 1$ and all $a_i > 0$, is not only a dominating measure but also equivalent to the family $\{\mu_{\varepsilon+L(x)} : x \in M\}$ (i.e. the measures in the family vanish on the same subsets as ν). Especially, this gives a necessary and sufficient condition for the domination of the probability measures $\mu_{\varepsilon+L(x)}$. In Section 5 we concentrate on special cases where μ_ε can be taken as ν . In these examples, we require that $\mu_{\varepsilon+L(x)}(U) = 0$ whenever $\mu_\varepsilon(U) = 0$ i.e. μ_ε is quasi-invariant with respect to translations with $L(x)$, where $x \in F$. This allows the use of any prior distribution on F . However, Remark 7 in Section 5 demonstrates that in dominated cases it is not always possible to choose μ_ε as ν .

We return to the question of partial uniqueness (Theorem 2.7). The conditions in the next theorems allow easier validation of the measurability and guarantee some continuity for the solutions. However, under the stronger assumption that the function $(x, z) \mapsto \rho(x, z)$ is jointly continuous and bounded, the solution μ is always \mathcal{F} -continuous on G (see Theorem 7.14.8 in [13]). Recall, that the class of all Souslin sets is quite large since all Borel subsets of a Souslin space are Souslin sets by Corollary 6.6.7 in [13].

Theorem 3.4. *Let $\mu_{\varepsilon+L(x)}$ be absolutely continuous with respect to a σ -finite measure ν for μ_X -almost every $x \in F$. If*

$$\rho(x, z) = \begin{cases} \frac{d\mu_{\varepsilon+L(x)}}{d\nu}(z) & \text{when } \mu_{\varepsilon+L(x)} \ll \nu \\ 0 & \text{otherwise.} \end{cases}$$

is a separately continuous function on some $F_0 \times A$, where F_0 is a Souslin subset of F with full μ_X -measure and A is a Souslin subset of G such that $\nu(A^C) = 0$, then ρ is $\mu_X \times \nu$ -measurable.

If additionally $\sup_{z \in K} \rho(x, z) \in L^1(\mu_X)$ for all compact sets $K \subset G$ then

$$(10) \quad z \mapsto \mu(U, z) = \frac{\int 1_U(x) \rho(x, z) d\mu_X(x)}{\int \rho(x, z) d\mu_X(x)}$$

is \mathcal{F} -continuous on $K \cap \{z \in A : 0 < \int \rho(x, z) d\mu_X(x) < \infty\}$ for every compact set $K \subset G$.

Proof. Assume that ρ is separately continuous. Since F_0 is a Souslin space, there exists a continuous surjection R from some complete separable metric space M onto F_0 . We consider first the function $(m, z) \mapsto \rho(R(m), z)$ on $M \times A$. This function is a pointwise limit of continuous functions due to a theorem of W. Rudin [124]. Hence the function $(m, z) \mapsto \rho(R(m), z)$ is $\mathcal{B}(M \times A) = \mathcal{B}(M) \otimes \mathcal{B}(A)$ -measurable. We compose it with a $\mu_X \times \nu$ -measurable mapping (R^{-1}, I) where the inverse comes from the measurable choice theorem (see Theorem 6.9.1 in [13], note that Souslin sets are universally measurable i.e. measurable with respect to any finite Radon measure by Theorem 7.4.1 in [13]). Then we see that $1_A(z)1_{F_0}(x)\rho(x, z)$, together with its equivalent mapping $\rho(x, z)$, is $\mu_X \times \nu$ -measurable.

By the Lebesgue dominated convergence theorem, we obtain the sequential continuity of the marginals. On Souslin spaces, the compact sets are metrizable (see Corollary 6.7.8 in [13]). In metrizable spaces sequential continuity coincides with continuity. \square

The above Theorem 3.4 shows \mathcal{F} -continuity of the solution μ on $\{z \in A : 0 < \int \rho(x, z) d\mu_X(x) < \infty\}$ when G is e.g. a k -space i.e. a subset C of G is closed if

and only if $C \cap K$ is closed for every compact $K \subset G$ (see Definition 43.8 in [157]). Indeed, it is well-known that a function f is continuous on a k -space if and only if it is continuous on every compact subset. In particular, this holds for all first-countable spaces, like metric spaces. Note, that the space of tempered distributions $\mathcal{S}'(\mathbf{R}^n)$ is a k -space when equipped with its strong topology but not with its weak topology, while the distribution space $\mathcal{D}'(U)$ is not a k -space with respect to either topology [61]. But $\mathcal{D}'(U)$ is a Lusin space [129] – i.e. a Hausdorff space that is a continuous injective image of a complete metric space – and can be equipped with a stronger metrizable topology inherited from the metric space. However, this topology depends on the chosen metric space and has all the drawbacks indicated in Remark 3.

We combine Theorem 2.7 and Theorem 3.4 in a simple case.

Corollary 1. *Let G be a k -space. Let $\mu_{\varepsilon+L(x)}$ be equivalent with a probability measure ν for every $x \in F$. Denote S_ν the topological support of ν . If*

$$\rho(x, z) = \frac{d\mu_{\varepsilon+L(x)}}{d\nu}(z)$$

is a separately continuous function on $F \times S_\nu$ and if $\sup_{z \in K} \rho(x, z) \in L^1(\mu_X)$ for all compact subsets $K \subset G$ then all solutions of the statistical inverse problem of estimating the distribution of X given Y that are \mathcal{F} -continuous on $\{z \in S_\nu : 0 < \int \rho(x, z) d\mu_X(x) < \infty\}$ coincide with

$$(11) \quad z \mapsto \mu(U, z) = \frac{\int 1_U(x) \rho(x, z) d\mu_X(x)}{\int \rho(x, z) d\mu_X(x)}$$

on $\mathcal{F} \times \{z \in S_\nu : 0 < \int \rho(x, z) d\mu_X(x) < \infty\}$.

The proof is an immediate consequence of the following lemma, where we characterize the topological support of μ_Y in more convenient terms.

Lemma 3.5. *Let $Y = L(X) + \varepsilon$, where X and ε are statistically independent. The topological support of μ_Y is the smallest closed set $S \subset G$ such that $\mu_{\varepsilon+L(x)}(S) = 1$ for μ_X -almost every $x \in F$. Moreover, if $\mu_{\varepsilon+L(x)}$ is equivalent with a probability measure ν for every $x \in F$, then the topological supports of μ_Y and ν coincide.*

Proof. The first claim follows from the convolution $\mu_Y(S) = \int_G \mu_{\varepsilon+L(x)}(S) d\mu_X(x)$. For the second claim, we note that $\mu_{\varepsilon+L(x)}(S) = \nu(S)$ for every closed set $S \subset G$ with full ν -measure (or full $\mu_{\varepsilon+L(x)}$ -measure). \square

4. CONVERGING APPROXIMATIONS

Throughout this section, we use the following assumptions

Definition 4.1. We say that Assumption A holds, if the following four conditions are satisfied.

- (1) Topological spaces F and G are locally convex Souslin topological vector spaces equipped with their Borel σ -algebras \mathcal{F} and \mathcal{G} , respectively.
- (2) The triple (Ω, Σ, P) is a complete probability space, X and X_n are F -valued random variables on Ω and ε is G -valued random variable on Ω . The random variables X and ε are independent. The random variables X_n and ε are independent.
- (3) The mapping $L : F \rightarrow G$ is continuous, and we denote $Y = L(X) + \varepsilon$ and $Y_n = L(X_n) + \varepsilon$.

- (4) The measure $\mu_{\varepsilon+L(x)}$ is absolutely continuous with respect to some σ -finite measure ν on (G, \mathcal{G}) for any $x \in F$, and its density

$$\rho(x, y) := \frac{d\mu_{\varepsilon+L(x)}}{d\nu}(y)$$

is a $\mu_Z \times \nu$ -measurable function on $F \times G$ for random variables $Z = X$ and $Z = X_n$, $n \in \mathbf{N}$.

When Assumptions A holds, we can use Theorem 3.3 to represent the approximated posterior distribution of X_n given $y = Y_n(\omega_0)$ as

$$(12) \quad \mu_n(U, y) := \frac{\int 1_U(x) \rho(x, y) d\mu_{X_n}(x)}{\int \rho(x, y) d\mu_{X_n}(x)}$$

and the posterior distribution of X given $y = Y(\omega_0)$ as

$$(13) \quad \mu(U, y) := \frac{\int 1_U(x) \rho(x, y) d\mu_X(x)}{\int \rho(x, y) d\mu_X(x)}$$

for all $U \in \mathcal{F}$ and $y \in M_0$, where

$$(14) M_0 = \{y \in G : 0 < \int \rho(x, y) d\mu_Z(x) < \infty, \text{ for } Z = X, X_n \text{ where } n \in \mathbf{N}\}$$

has full μ_Y -measure.

We recall some definitions on the convergence of measures.

Definition 4.2. Let m and m_n , where $n \in \mathbf{N}$, be σ -finite measures on a topological space F equipped with the Borel σ -algebra \mathcal{F} .

- (i) The measures m_n converge weakly to m if

$$\lim_{n \rightarrow \infty} \int f(x) dm_n(x) = \int f(x) dm(x)$$

for all bounded continuous functions f on F .

- (ii) The measures m_n converge setwise to m if

$$\lim_{n \rightarrow \infty} m_n(U) = m(U)$$

for every $U \in \mathcal{F}$.

- (iii) The measures m_n converge in variation to m if

$$\lim_{n \rightarrow \infty} \sup_{U \in \mathcal{F}} |m_n(U) - m(U)| = 0.$$

It is well-known that the weak convergence of the probability measures implies the convergence of certain expectations on regular enough spaces. The following theorem generalizes slightly Lemma 8.4.3 in [13] by requiring that the discontinuities of f belong to some m -zero measurable set. The proof for the present case seems no to be readily available in the literature.

Lemma 4.3. Let F be a locally convex Souslin topological vector space and m, m_n , where $n \in \mathbf{N}$, be finite measures on (F, \mathcal{F}) . Let f be an m -integrable Borel function on F whose discontinuities are contained in an m -zero measurable set. If

$$\lim_{C \rightarrow \infty} \sup_n \int_{|f| > C} |f|(x) dm_n(x) = 0,$$

then $m_n(f)$ converge to $m(f)$ whenever m_n converge weakly to m

Proof. Consider first a bounded Borel measurable function g , say $|g| \leq c$, whose points of discontinuity belong to an m -zero measurable set N_g . The integral of g can be written as

$$\int g dm_n = \int_{-c}^c t d(m_n \circ g^{-1})(t)$$

where the integrand is bounded and continuous on $[-c, c]$. We show that the measures $m_n \circ g^{-1}$ on $[-c, c]$ converge weakly to $m \circ g^{-1}$, which immediately implies the convergence of $m_n(g)$ to $m(g)$ as n grows. We apply a well-known property of completely regular spaces (see Corollary 8.2.4 in [13]), according to which the weak convergence of $m_n \circ g^{-1}$ to the Radon measure $m \circ g^{-1}$ is equivalent to

$$\limsup_{n \rightarrow \infty} m_n \circ g^{-1}(A) \leq m \circ g^{-1}(A)$$

for all closed sets A . Note that all locally convex spaces are completely regular. If $A \subset [-c, c]$ is closed, the closure $\overline{g^{-1}(A)} \subset g^{-1}(A) \cup N_g$ because g is continuous in the relative topology of $G \setminus N_g$. Since N_g has zero m -measure,

$$m(g^{-1}(A)) = m(\overline{g^{-1}(A)}) \geq \limsup_{n \rightarrow \infty} m_n(\overline{g^{-1}(A)}) \geq \limsup_{n \rightarrow \infty} m_n(g^{-1}(A))$$

by the weak convergence of measures m_n .

Let \wedge denote the binary operation of taking the minimum of two real numbers. For the general case, we approximate f with bounded functions $\text{sgn}(f)(|f| \wedge C)$ in the difference

$$\begin{aligned} |(m_n - m)(f)| &= |(m_n - m)(f - \text{sgn}(f)(|f| \wedge C) + \text{sgn}(f)(|f| \wedge C))| \\ &\leq \sup_n \int_{|f| > C} |f|(x) d(m_n + m) + \left| \int \text{sgn}(f)(|f| \wedge C) d(m_n - m) \right|. \end{aligned}$$

By the assumption, the first term in the sum (15) gets arbitrarily small when C is chosen large enough. Since $\text{sgn}(f)(|f| \wedge C) =: g$ is bounded and m -a.e. continuous, the second term in the sum (15) converge to zero for fixed C when n grows by the weak convergence of the measures m_n . \square

Lemma 4.3 can be applied for $f = g\rho(\cdot, y)$, $m_n = \mu_{X_n}$ and $m = \mu_X$, where g is any continuous bounded function on F .

Theorem 4.4. *Let Assumption A hold and let μ_n, μ and M_0 be defined by equations (12), (13) and (14), respectively. Let $y \in M_0$ and let the discontinuities of $x \mapsto \rho(x, y)$ belong to a μ_X -zero measurable set. If the functions $x \mapsto \rho(x, y)$ satisfy*

$$\lim_{C \rightarrow \infty} \sup_n \int_{|\rho(\cdot, y)| > C} |\rho(x, y)| d\mu_{X_n}(x) = 0.$$

then the approximated posterior distributions $\mu_n(\cdot, y)$ converge weakly to the posterior distribution $\mu(\cdot, y)$ whenever the approximated prior distributions μ_{X_n} converge weakly to the prior distribution μ_X .

The conditional mean is a common estimate for the unknown. The convergence of conditional mean estimates in the weak topology of F is considered next.

Theorem 4.5. *Let Assumption A hold, let $y \in M_0$ and let the discontinuities of $x \mapsto \rho(x, y)$ belong to a μ_X -zero measurable set. If functions $x \mapsto \langle x, \alpha \rangle^k \rho(x, y)$ belong to $L^1(\mu_X)$ and satisfy*

$$\lim_{C \rightarrow \infty} \sup_n \int_{|\langle \cdot, \alpha \rangle|^k \rho(\cdot, y) > C} |\langle x, \alpha \rangle|^k \rho(x, y) d\mu_{X_n}(x) = 0,$$

for $k = 0, 1$ and $\alpha \in F'$, then the approximated weak conditional mean estimates $\mu_n(\langle \cdot, \alpha \rangle, y)$ converge to the weak conditional mean estimate $\mu(\langle \cdot, \alpha \rangle, y)$ whenever the approximated prior distributions μ_{X_n} converge weakly to the prior distribution μ_X .

Proof. The nominators and the denominators of

$$\mu_n(\langle \cdot, \alpha \rangle, y) = \frac{\int \langle x, \alpha \rangle \rho(x, y) d\mu_{X_n}(x)}{\int \rho(x, y) d\mu_{X_n}(x)}$$

converge as n grows by Lemma 4.3, and the limit of their quotients is $\mu(\langle \cdot, \alpha \rangle, y)$. \square

When F is a separable Banach space, we can state conditions for the norm convergence of the conditional mean estimates that are defined as Bochner integrals, that is, $\int x dm(x)$ is taken to be the limit in F of integrals of simple functions of the form $x \mapsto \sum_{i=1}^n x_{k_i} 1_{U_{k_i}}(x)$, where $x_{k_i} \in F$ and $U_{k_i} \in \mathcal{F}$.

Theorem 4.6. *Let Assumption A hold and let μ_n, μ and M_0 be defined by equations (12), (13) and (14), respectively. Let $y \in M_0$ and let the discontinuities of $x \mapsto \rho(x, y)$ belong to a μ_X -zero measurable set. Additionally, let F be a separable Banach space with norm $\|\cdot\|$.*

If $\|\cdot\| \rho(\cdot, y) \in L^1(\mu_X)$ satisfies

$$(16) \quad \lim_{C \rightarrow \infty} \sup_n \int_{\{x: \|x\|^k \rho(x, y) > C\}} \|x\|^k \rho(x, y) d\mu_{X_n}(x) = 0$$

for $k = 0, 1$ then the conditional mean estimates $\int x \mu_n(dx, y)$ converge in the norm of F to the conditional mean estimate $\int x \mu(dx, y)$ whenever the approximated prior distributions μ_{X_n} converge weakly to the prior distribution μ_X .

Proof. The assumptions for $k = 0$ guarantee that the denominators $\int \rho(x, y) d\mu_{X_n}(x)$ of the posterior distributions converge to $\int \rho(x, y) d\mu_X(x)$ as $n \rightarrow \infty$ by Lemma 4.3.

The function $\|\cdot\| \rho(\cdot, y)$ has a finite expectation with respect to all measures μ_{X_n} , $n \in \mathbb{N}$ and μ_X . Therefore, the mapping $x \mapsto x \rho(x, y)$ is Bochner integrable with respect to all μ_{X_n} and μ_X , and its discontinuities belong to a μ_X -zero measurable set.

For the moment, let us choose random variables X_n and X on another probability space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ in such a way that their image measures are μ_{X_n} and μ_X , respectively, and the random variables X_n converge almost surely to X as $n \rightarrow \infty$. Such a choice is possible by the Skorokhod representation theorem (see Theorem 8.5.4 in [13]). Especially, $X_n \rho(X_n, y) - X \rho(X, y)$ is Bochner integrable with respect to the probability measure \tilde{P} by the triangle inequality. Denote with $A_C = \{\|X_n \rho(X_n, y) - X \rho(X, y)\| > C\}$ for $C > 0$. Then the nominators of the posterior distributions satisfy

$$\begin{aligned} \left\| \int x \rho(x, y) d\mu_{X_n}(x) - \int x \rho(x, y) d\mu_X(x) \right\| &= \left\| \int X_n \rho(X_n, y) - X \rho(X, y) d\tilde{P} \right\| \\ &\leq \int_{A_C} \|X_n \rho(X_n, y) - X \rho(X, y)\| d\tilde{P} + \\ &\quad \int C \wedge \|X_n \rho(X_n, y) - X \rho(X, y)\| d\tilde{P} \\ &=: I_1(n; C) + I_2(n; C). \end{aligned}$$

The integrals $I_1(n; C)$ vanish when $C \rightarrow \infty$ since their arguments are uniformly integrable. Indeed, both $\|X_n \rho(X_n, y)\|$ and $\|X \rho(X, y)\|$ are uniformly integrable, as is also their sum. Any sequence of non-negative functions that has uniformly integrable upper bound with respect to a finite measure is again uniformly integrable with respect to the finite measure. These facts are direct consequences from the characterization of the uniformly integrable function through uniformly absolutely continuous integrals (see Proposition 4.5.3 in [13]).

The integrals $I_2(n; C)$ for a fixed C converge to zero as $n \rightarrow \infty$ by the Lebesgue dominated convergence theorem and continuity properties of ρ . \square

Remark 6. Theorem 4.6 generalizes the similar convergence results of [63, 96], in which the spaces F and G are separable Banach spaces, ε is Gaussian for non-Gaussian ε on more general spaces. In [63, 96], it is assumed that

$$\sup_n \mathbf{E}[\exp(a\|X_n\|)] < \infty$$

for all $a > 0$. This attractive condition is stronger than (16) for the given ρ , which has the form $\rho(x, y) = \exp(\langle y, Lx \rangle_G - \frac{1}{2}\|Lx\|_H^2)$, where H is a certain Hilbert space. Indeed, by the de la Vallée Poussin theorem (e.g. Theorem 4.5.9 in [13]) the condition (16) is equivalent to the existence of a nonnegative increasing functions g_k on \mathbf{R} such that $\lim_{t \rightarrow \infty} t^{-1}g_k(t) = +\infty$ and $\sup_n \int g_k(|x|^k \rho(x, y)) d\mu_{X_n}(x) < \infty$. Moreover,

$$\|x\|^k \rho(x, y) = \|x\|^k \exp(\langle y, Lx \rangle_G - \frac{1}{2}\|Lx\|_H^2) \leq \exp(a\|x\|),$$

where $a = 1 + \|y\|_G \|L\|_{F \rightarrow G}$. The choice $g(t) = t^2$ guarantees that the condition (16) holds when $\sup_n \mathbf{E}[\exp(a\|X_n\|)] < \infty$.

Next, we pursue after a stronger convergence of the posteriors.

Theorem 4.7. *Let Assumption A hold and let μ_n , μ and M_0 be defined by equations (12), (13) and (14), respectively. Let $y \in M_0$.*

If the measures $U \mapsto \int_U \rho(x, y) d\mu_n(x)$ on (F, \mathcal{F}) are uniformly bounded, and equicontinuous at zero in the sense that for every decreasing sequence $\{U_i\} \subset \mathcal{F}$ with empty intersection,

$$\lim_{i \rightarrow \infty} \sup_n \int_{U_i} \rho(x, y) d\mu_n(x) = 0,$$

then the approximated posterior distributions $\mu_n(\cdot, y)$ converge setwise (or in variation) to the posterior distribution $\mu(\cdot, y)$ whenever the approximated prior distributions μ_{X_n} converge setwise (or in variation) to the prior distribution μ_X .

Proof. Assume first, that the approximated prior distributions converge setwise. Define a finite measure $\nu := \mu_X + \sum_{n=1}^{\infty} 2^{-n} \mu_{X_n}$ on (F, \mathcal{F}) . Each μ_{X_n} is absolutely continuous with respect to ν and has Radon-Nikodym density $f_n := \frac{d\mu_{X_n}}{d\nu}$.

The measurable function $x \mapsto \rho(x, y)$ is an increasing limit of some simple functions $\phi_y^{(i)}(x)$ and, by Egorov's theorem, the convergence is almost uniform with respect to the measure ν . That is, for every $\varepsilon > 0$, there exist a set A_ε such that $\phi_y^{(i)}$ converge uniformly to $\rho(\cdot, y)$ on A_ε and $\nu(A_\varepsilon^c) < \varepsilon$. One may choose a sequence $\varepsilon_j \rightarrow 0$ and get increasing sets A_{ε_j} such that $\nu(\cap_j A_{\varepsilon_j}^c) = 0$ and the simple functions

$\phi_y^{(i)}$ converge uniformly on each A_{ε_j} . But then

$$\begin{aligned} \left| \int_U \rho(x, y) d\mu_X(x) - \int_U \rho(x, y) d\mu_{X_n}(x) \right| &\leq \left| \int_U \phi_y^{(i)}(x) d\mu_X(x) - \int_U \phi_y^{(i)}(x) d\mu_{X_n}(x) \right| \\ &\quad + \left| \int_U \rho(x, y) - \phi_y^{(i)}(x) d\mu_X(x) - \int_U \rho(x, y) - \phi_y^{(i)}(x) d\mu_{X_n}(x) \right|, \end{aligned}$$

where the last term is bounded by

$$\begin{aligned} \int_{U \cap A_{\varepsilon_j}} |\rho(x, y) - \phi_y^{(i)}(x)| (f_n + f)(x) d\nu(x) &+ \int_{A_{\varepsilon_j}^C} |\rho(x, y) - \phi_y^{(i)}(x)| (f_n + f)(x) d\nu(x) \\ &=: I_1 + I_2 \end{aligned}$$

In the integral I_2 , the estimate $|\rho(x, y) - \phi_y^{(i)}(x)| \leq \rho(x, y)$ gives

$$(17) \quad I_2 \leq \sup_n \int_{A_{\varepsilon_j}^C} \rho(x, y) d(\mu_{X_n} + \mu_X)(x).$$

If the intersection of the sets $A_{\varepsilon_j}^C$ is not empty, we subtract the ν -zero measurable intersection from each $A_{\varepsilon_j}^C$. Then the equicontinuity at zero of measures $U \mapsto \int_U \rho(x, y) d\mu_{X_n}(x)$ implies that the integrals (17) are bounded by any given positive number when j is large enough. The final thing is to choose the simple function $\phi_y^{(i)}$ so that $|\rho(\cdot, y) - \phi_y^{(i)}|$ is small enough on chosen A_{ε_j} and then choose large enough n so that $|\mu_X(1_U \phi_y^{(i)}) - \mu_{X_n}(1_U \phi_y^{(i)})|$ gets small enough. This is possible since the integrand is a bounded simple function and μ_{X_n} converge setwise to μ_X .

In order to prove convergence in variation, just add $\sup_{U \in \mathcal{F}}$ in front of the above estimates. \square

Equivalent conditions for the equicontinuity at zero of a bounded family of measures m_n on (F, \mathcal{F}) are presented in Lemma 4.6.5 in [13]. The setwise convergence of measures μ_{X_n} actually implies that they are equicontinuous at zero by Theorem 4.6.3 in [13].

5. EXAMPLES OF NOISE

Below, some cases are presented, where the Radon-Nikodym derivatives

$$\frac{d\mu_{\varepsilon+L(x)}}{d\nu}$$

exist with respect to some σ -finite measure ν . Two first cases, where the noise term is finite-dimensional or Gaussian, are well-known. For these cases, we apply the results of previous sections. The next four cases demonstrate that the approach taken in this paper applies also for more general noise models.

5.1. Finite-dimensional noise with a probability density. This example extends the convergence results in [44] to locally convex Souslin space-valued unknowns. Let G be the Euclidian space \mathbf{R}^k , let F be a locally convex Souslin space, and let $L : F \rightarrow G$ be a continuous mapping. Consider the statistical inverse problem of estimating the distribution of an F -valued random variable X given a sample y of a G -valued random variable $Y = L(X) + \varepsilon$, where the G -valued random variable ε is statistically independent from X . In order to use the representation formula of Theorem 3.3 for the essentially unique posterior distribution of X given

a sample y_0 of Y , we need the required σ -finite measure ν . A natural choice is to take the Lebesgue measure as ν , when possible.

Assume that the noise ε is \mathbf{R}^k -valued random vector whose image measure μ_ε is absolutely continuous with respect to the Lebesgue measure, say $\mu_\varepsilon(dx) = D_\varepsilon(x)dx$, with the property that $D_\varepsilon > 0$ almost everywhere. Especially, μ_ε is then equivalent to the Lebesgue measure.

In Theorem 3.3, the Radon-Nikodym derivative of $\mu_{\varepsilon+L(x)}$ with respect to the Lebesgue measure, i.e. $(x, y) \mapsto D_\varepsilon(y - L(x))$, is required to be jointly measurable. Since D_ε is measurable, and the addition is measurable, the continuity of L suffices here. We obtain an essentially unique solution μ of the statistical inverse problem of estimating the distribution of X given a sample y_0 of Y that satisfies

$$\mu(U, y_0) = \frac{\int_U D_\varepsilon(y_0 - L(x)) d\mu_X(x)}{\int D_\varepsilon(y_0 - L(x)) d\mu_X(x)}$$

for all $U \in \mathcal{F}$ and all y_0 such that $0 < \int D_\varepsilon(y_0 - L(x)) d\mu_X(x) < \infty$. Here $D_\varepsilon(y_0 - L(x))$ is often called the likelihood function. If D_ε is continuous and bounded, we may drop out the word "essentially", as the solution is the unique continuous solution by Corollary 1 (the topological support of μ_Y is the whole space by Lemma 3.5 since $\mu_{\varepsilon+L(x)}$ is equivalent with the Lebesgue measure).

When X is an \mathbf{R}^m -valued random variable with a density $D_{pr}(x)$ with respect to the Lebesgue measure, we get the familiar expression

$$\mu(U, y) = \frac{\int_U D_\varepsilon(y - L(x)) D_{pr}(x) dx}{\int D_\varepsilon(y - L(x)) D_{pr}(x) dx}$$

for all y such that $0 < \int D_\varepsilon(y - L(x)) D_{pr}(x) dx < \infty$.

Remark 7. When $D_\varepsilon \geq 0$ almost everywhere, μ_ε need not be equivalent to the Lebesgue measure. Moreover, the translated measure $\mu_{\varepsilon+L(x)}$ need not be absolutely continuous with respect to μ_ε .

We consider next the convergence of posterior distributions. Let μ_{X_n} be the finite-dimensional distributions that approximate μ_X and denote with X_n the corresponding F -valued random variables that are statistically independent from ε . Denote

$$\mu_n(U, y) = \frac{\int_U D_\varepsilon(y - L(x)) d\mu_{X_n}(x)}{\int D_\varepsilon(y - L(x)) d\mu_{X_n}(x)}$$

the corresponding solutions of estimating the probabilities of X_n given $Y_n = L(X_n) + \varepsilon$. When D_ε is continuous and bounded, the probabilities $\mu_n(\cdot, y)$ converge weakly to

$$\mu(\cdot, y) = \frac{\int D_\varepsilon(y - L(x)) d\mu_X(x)}{\int D_\varepsilon(y - L(x)) d\mu_X(x)}$$

for all y such that

$$\inf_n \int D_\varepsilon(y - L(x)) d\mu_{X_n}(x) > 0 \text{ and } \sup_n \int D_\varepsilon(y - L(x)) d\mu_{X_n}(x) < \infty$$

whenever μ_{X_n} converge weakly to μ_X by Theorem 4.4. Also Theorem 4.6 and Theorem 4.7 are available, provided the assumptions hold.

In practical applications one often takes such approximations of μ_X that can be identified with a probability distribution $D_{pr}^{(n)} dx$ on \mathbf{R}^n by some linear isomorphism \mathcal{I}_n defined on a subspace of full measure i.e. $\mu_{X_n} \circ \mathcal{I}_n^{-1} = D_{pr}^{(n)} dx$.

5.2. Infinite-dimensional Gaussian noise. The finite-dimensional Gaussian noise model is often chosen because of its relatively straightforward justification – if the total noise is produced by many identical independent noise sources, the sum is nearly Gaussian by the central limit theorem. For instance, this applies to the origin of thermal noise in electrical circuits, where heat motion of the charge carriers disturbs the analog signal. The usual model of thermal noise is white Gaussian noise, which is an acceptable approximation on usual frequencies.

We first recall a method for constructing infinite-dimensional Gaussian random vectors by a procedure linked to abstract Wiener spaces [12].

5.2.1. Basics of Hilbert space-valued Gaussian random variables. Let H be a separable Hilbert space. We define Z as a random sum

$$(18) \quad Z = \sum_{i=1}^{\infty} Z_i e_i,$$

where Z_i are independent standard normal random variables on (Ω, Σ, P) and $\{e_i\}$ is an orthonormal basis of H . Clearly, the sum does not converge a.s. in H . Instead, we take a larger Hilbert space G into which H can be imbedded with an injective Hilbert-Schmidt operator j . When the range of the imbedding is dense, the triple (j, H, G) is a special case of an abstract Wiener space [12]. However, we do not require the range to be dense. Let G' denote the dual space of G and $\langle \cdot, \cdot \rangle$ the duality between G and G' .

A sufficient condition for the a.s. convergence of the random sums $\sum_{i=1}^n Z_i e_i$ in G is that the series

$$\sum_{i=1}^{\infty} \mathbf{E}[\|Z_i e_i\|_G^2] = \sum_{i=1}^{\infty} \|e_i\|_G^2.$$

is convergent [73]. But this follows from the Hilbert-Schmidt property of the inclusion map j .

Since G is a separable Fréchet space (more generally, a locally convex Souslin space [129]), its Borel σ -algebras with respect to the weak and the original topology coincide. The benefit of the weak topology is that the measurability of the limit $Z = \lim_{n \rightarrow \infty} \sum_{i=1}^n Z_i e_i$ can be checked similarly as in the case of real-valued functions with sets of the type $\cap_{i=1}^k \{|\langle Z, \phi_i \rangle - a_i| < c_i\}$. We conclude that the a.s. limit Z of the random sums defines a measurable mapping from (Ω, Σ, P) to (G, \mathcal{G}) . Its image measure $\mu_Z = P \circ Z^{-1}$ can be viewed also as a countably additive cylinder set measure.

In general, the mean of a random variable Z is the vector $m \in G'' = G$ such that $\mathbf{E}[\langle Z, \phi \rangle] = \langle m, \phi \rangle$ for all $\phi \in G'$ and the covariance operator of Z is the mapping $C : G' \rightarrow G$ such that

$$\langle C\phi, \psi \rangle = \mathbf{E}[(\langle Z, \phi \rangle - \langle m, \phi \rangle)(\langle Z, \psi \rangle - \langle m, \psi \rangle)]$$

for all $\phi, \psi \in G'$ [12].

Since limits of Gaussian random variables are Gaussian, the random variable

$$\langle Z + m, \phi \rangle = \langle m, \phi \rangle + \sum_{i=1}^{\infty} Z_i \langle e_i, \phi \rangle,$$

where $m \in G$, has a characteristic function

$$(19) \quad e^{i\langle m, \phi \rangle - \frac{1}{2}\langle C\phi, \phi \rangle} = \mathbf{E}[e^{i\langle Z, \phi \rangle}]$$

for all $\phi \in G'$. In our case, the random variable Z has mean $m = 0$ and covariance $\langle C\phi, \phi \rangle = \sum_{i=1}^{\infty} \langle e_i, \phi \rangle^2$. The covariance $\langle C\phi, \phi \rangle$ is the squared norm of ϕ in the strong dual space H' of H . Indeed, the linear form $\langle j \cdot, \phi \rangle_{G, G'}$ is continuous on H so it belongs to H' and its norm is $\langle C\phi, \phi \rangle$. For short, we denote $\phi \in H'$. The covariance $\langle C\phi, \phi \rangle$ for any $\phi \in G'$ is finite, since $H \hookrightarrow G$ implies that $G' \hookrightarrow H'$ continuously. The dual space G' is actually dense in H' as a consequence of the Hahn-Banach theorem. Indeed, if $h'_0 \in H' \setminus \overline{j'(G')} \neq \emptyset$ then there would exist $h \in H'' = H$ such that $\langle h, h'_0 \rangle = 1$ and $\langle h, h' \rangle = 0$ for every $h' \in j'(G')$. But $j'(G')$ separates the points in H because of the injectivity of j . Therefore, $h = 0$ and hence $j'(G')$ is dense in H' .

The mapping $G' \ni \phi \mapsto \langle C\phi, \phi \rangle$ has an extension $H' \ni g \mapsto \langle \bar{C}g, g \rangle := \|g\|_{H'}^2$. By the polarization equality, \bar{C} is the isometric isomorphism between H' and H defined by the Riesz representation theorem. We continue to denote \bar{C} with C .

Remark 8. It is well-known that the sample space G of Z can be replaced with any bigger locally convex Souslin vector space G_0 into which G can be continuously and injectively embedded. For example, G_0 may be the distribution space $\mathcal{D}'(U)$, where $U \subset \mathbf{R}^n$ is open, equipped with the usual weak topology.

Measures having characteristic functions of the above form (19) are called Gaussian measures. Especially, the image measure $\mu_Z = P \circ Z^{-1}$ is Gaussian. Random variables, whose image measures are Gaussian, are called Gaussian random variables. The space H is the so-called *Cameron-Martin space* of μ_Z .

By Theorems 3.2.3, 3.2.7 and 3.5.1 in [12] any zero-mean Gaussian random variable on a locally convex Souslin space is equivalent with a random variable of the form (18). More details on Gaussian measures can be found in [12, 55, 91].

5.2.2. Inverse problems with Gaussian noise. We consider the statistical inverse problem of estimating the distribution of X given a sample of $Y = L(X) + \varepsilon$, where ε is a zero mean Gaussian random variable that has values in a separable Hilbert space G .

We denote with H_{μ_ε} the Cameron-Martin space of μ_ε and with $C_\varepsilon : H'_{\mu_\varepsilon} \rightarrow H_{\mu_\varepsilon}$ the covariance operator of ε . The unknown random variable X has values in some locally convex Souslin topological vector space F . The random variables ε and X are taken to be independent. The direct theory $L : F \rightarrow G$ is a continuous mapping that satisfies the following additional restrictive conditions: $L : F \rightarrow H_{\mu_\varepsilon}$ is continuous, the range of the combined mapping $C_\varepsilon^{-1}L$ belongs to G' where G' is the strong dual of G , and the mapping $C_\varepsilon^{-1}L : F \rightarrow G'$ is continuous.

As an approximated model, we take a sequence of F -valued random variables X_n that satisfy the same conditions as X . We denote $Y_n := L(X_n) + \varepsilon$.

Recalling Remark 6, we require that

$$(20) \quad \mathbf{E} \left[e^{a\|L(X)\|_{G'}} \right] \wedge \sup_n \mathbf{E} \left[e^{a\|L(X_n)\|_{G'}} \right] < \infty$$

for all $a > 0$. The condition holds especially when the range of $C_\varepsilon^{-1}L$ is bounded in G' .

According to the famous Cameron-Martin formula (see Corollary 2.4.3 and Theorem 3.2.3 in [12]), the Gaussian measures μ_ε and $\mu_{\varepsilon+L(x)}$ are equivalent when $L(x) \in H_{\mu_\varepsilon}$. The corresponding Radon-Nikodym density is

$$\rho(x, z) := \frac{d\mu_{\varepsilon+L(x)}}{d\mu_\varepsilon}(z) = \exp \left(\langle z, C_\varepsilon^{-1}L(x) \rangle - \frac{1}{2} \|L(x)\|_{H_{\mu_\varepsilon}}^2 \right), \quad z \in G.$$

Remark 9. In the Cameron-Martin formula, the notation $\langle \cdot, \cdot \rangle$ is, in general, a measurable extension of the duality. Namely, the vector $C_\varepsilon^{-1}L(x)$ need not belong to the space G' but in the larger space H'_{μ_ε} . But G' is dense in H'_{μ_ε} . Following Lemma 2.2.8. in [12], we may define $\langle z, C_\varepsilon^{-1}L(x) \rangle$ as the limit of $\langle z, \phi_n \rangle$ in $L^2(\mu_\varepsilon)$ where $\phi_n \in G'$ converge to $C_\varepsilon^{-1}L(x)$ in H'_{μ_ε} as $n \rightarrow \infty$. Especially, $\langle z, C_\varepsilon^{-1}L(x) \rangle$ is a Gaussian random variable on $(G, \mathcal{G}, \mu_\varepsilon)$. Different approximating sequences lead to equivalent random variables, since the limits coincide in $L^2(\mu_\varepsilon)$.

When the range $C_\varepsilon^{-1}L \subset G'$, we have $\langle z, C_\varepsilon^{-1}L(x) \rangle = \langle z, C^{-1}\varepsilon \rangle_{G, G'}$, and, consequently, the Radon-Nikodym density is separately continuous with respect to z on G and with respect to x on F . By Theorem 3.4, ρ is $\mu_X \times \mu_\varepsilon$ -measurable. In Theorem 3.3, we may choose $\nu = \mu_\varepsilon$ and take

$$\mu(U, y) := \frac{\int_U \exp(\langle y, C_\varepsilon^{-1}L(x) \rangle - \frac{1}{2}\|L(x)\|_{H_\varepsilon}^2) d\mu_X(x)}{\int \exp(\langle y, C_\varepsilon^{-1}L(x) \rangle - \frac{1}{2}\|L(x)\|_{H_\varepsilon}^2) d\mu_X(x)}$$

as an essentially unique solution for all $y \in G$. Note, that our assumptions guarantee that

$$0 < \exp\left(\langle y, C_\varepsilon^{-1}L(x) \rangle - \frac{1}{2}\|L(x)\|_{H_\varepsilon}^2\right) \leq \exp(\|y\|_G \|C_\varepsilon^{-1}L(x)\|_{G'}) \in L^1(\mu_X)$$

so that the set M_0 in (14) is empty. Similarly, when X_n satisfies the same conditions as X , we obtain

$$\mu_n(U, y) := \frac{\int_U \exp(\langle y, C_\varepsilon^{-1}L(x) \rangle - \frac{1}{2}\|L(x)\|_{H_\varepsilon}^2) d\mu_{X_n}(x)}{\int \exp(\langle y, C_\varepsilon^{-1}L(x) \rangle - \frac{1}{2}\|L(x)\|_{H_\varepsilon}^2) d\mu_{X_n}(x)}$$

for all $y \in G$.

We consider next the partial uniqueness of the solutions μ and μ_n on $\mathcal{F} \otimes G$. Denote with S_{μ_ε} the support of μ_ε on G , which coincides with the closure of the Cameron-Martin space H_{μ_ε} in G by Theorem 3.6.1 in [12]. The measure $\mu_{\varepsilon+L(x)}$ is equivalent with μ_ε by the Cameron-Martin formula. Hence, the measures μ_Y and μ_{Y_n} have the same topological support as the measure μ_ε by Lemma 3.5. We conclude that $S_{\mu_Y} = S_{\mu_{Y_n}} = \overline{H_{\mu_\varepsilon}}$. Since $\sup_{z \in K} \rho(x, z) \leq \exp(C\|C_\varepsilon^{-1}L(x)\|_{G'})$, the solutions μ and μ_n are \mathcal{F} -continuous on $G \cap \overline{H_{\mu_\varepsilon}}$ by Theorem 3.4. Hence, μ and μ_n are the only \mathcal{F} -continuous solutions on $G \cap \overline{H_{\mu_\varepsilon}}$ by Corollary 1. In the light of Corollary 1 and the discussion preceding it, the partial uniqueness is not so simple in the situation described in Remark 8.

In order to apply Theorem 4.4, we use the continuity of $x \mapsto \rho(x, y)$ and the uniform integrability that follows from the assumption (20). Consequently, Theorem 4.4 holds. If, for example, the range of $C_\varepsilon L$ is a bounded set in G' , also Theorem 4.7 is available.

Remark 10. In general, the measure $\mu_Y = \mu_{L(X)+\varepsilon}$ does not satisfy $\mu_{\varepsilon+L(x)} \ll \mu_Y$ for μ_X -almost every x . Indeed, take X and ε to be independent Gaussian random variables with the same Cameron-Martin space $L^2(I)$, where I is the unit interval $(0, 1)$. Let L be the identity. If $\mu_Y(U) = 0$, then $\mu_{\varepsilon+x}(U) = 0$ for μ_X -almost every x by the formula

$$(21) \quad \mu_Y(U) = \mathbf{E}[\mathbf{E}1_Y(U)|X]] = \int \mu_{\varepsilon+x}(U) d\mu_X(x).$$

Suppose that $\mu_{\varepsilon+x} \ll \mu_Y$ for μ_X -a.e. x , say for all $x \in M$ such that $\mu_X(M) = 1$. The random variable $Y = X + \varepsilon$ is also Gaussian, and any two Gaussian measures

on the same locally convex space are either equivalent or singular. But then $\mu_{\varepsilon+x_1}$ is equivalent to μ_Y and $\mu_{\varepsilon+x_2}$ is equivalent to μ_Y for any $x_1, x_2 \in M$ so also the two measures $\mu_{\varepsilon+x_1}$ and $\mu_{\varepsilon+x_2}$ are equivalent. But $P(X \in L^2(I)) = 0$ so equivalence should hold also for some $x_1, x_2 \notin L^2(I)$, which is impossible by the Cameron-Martin theorem. The μ_X -zero measurable set in (21) necessarily depends on U in this case.

5.3. Gaussian dominated noise. We consider a simple modification of Gaussian noise. Suppose that the assumptions in Section 5.2.2 hold except that instead of $Y = L(X) + \varepsilon$ we are observing $\tilde{Y} = L(X) + \tilde{\varepsilon}$, where $\mu_{\tilde{\varepsilon}}$ is dominated by the Gaussian measure μ_{ε} i.e.

$$\frac{d\mu_{\tilde{\varepsilon}}}{d\mu_{\varepsilon}}(y) = f(y)$$

for some $f \in L^1(\mu_{\varepsilon})$. The translation of $\mu_{\tilde{\varepsilon}}$ by $L(x)$ has the form

$$\begin{aligned} \mu_{\tilde{\varepsilon}+L(x)}(V) &= \int 1_V(y + L(x)) d\mu_{\tilde{\varepsilon}}(y) \\ &= \int 1_V(y + L(x)) f(y) d\mu_{\varepsilon}(y) \\ &= \int_V f(y - L(x)) d\mu_{\varepsilon+L(x)}(y) \\ &= \int_V f(y - L(x)) \exp\left(\langle y, C_{\varepsilon}^{-1}L(x) \rangle - \frac{1}{2}\|L(x)\|_{H_{\mu_{\varepsilon}}}^2\right) d\mu_{\varepsilon}(y). \end{aligned}$$

The integrand is a $\mu_X \times \mu_{\varepsilon}$ -measurable functions as a product of two $\mu_X \times \mu_{\varepsilon}$ -measurable functions. By Theorem 3.3, the posterior distribution of X given a sample y of $\tilde{Y} = L(X) + \tilde{\varepsilon}$ can be taken to be

$$(22) \quad \mu(U, y) = \frac{\int_U f(y - L(x)) \exp\left(\langle y, C_{\varepsilon}^{-1}L(x) \rangle - \frac{1}{2}\|L(x)\|_{H_{\mu_{\varepsilon}}}^2\right) d\mu_X(x)}{\int f(y - L(x)) \exp\left(\langle y, C_{\varepsilon}^{-1}L(x) \rangle - \frac{1}{2}\|L(x)\|_{H_{\mu_{\varepsilon}}}^2\right) d\mu_X(x)}$$

whenever the denominator is positive.

For instance, let $\tilde{\varepsilon}$ to be a restriction of ε to some open set $K \in \mathcal{G}$ that has positive μ_{ε} -measure. This means that the noise $\tilde{\varepsilon} = \varepsilon|_K$ has the distribution

$$(23) \quad \mu_{\tilde{\varepsilon}}(V) = \frac{P(\varepsilon \in K \cap V)}{P(\varepsilon \in K)}$$

for all $V \in \mathcal{G}$ i.e. we consider conditional probabilities

$$\mu_{\tilde{\varepsilon}}(V) = \mu_{\varepsilon}(V|K).$$

Note that as a Borel set, K is of the form $K = \{y \in G : (\langle y, \phi_1 \rangle, \langle y, \phi_2 \rangle, \dots) \in E\}$, where $\phi_i \in G'$ separate the points in G and $E \in \mathcal{B}(\mathbf{R}^{\infty})$. The Radon-Nikodym density of $\mu_{\tilde{\varepsilon}}$ with respect to μ_{ε} is by (23)

$$f(y) = \frac{d\mu_{\tilde{\varepsilon}}}{d\mu_{\varepsilon}}(y) = \frac{1}{\mu_{\varepsilon}(K)} 1_K(y).$$

By Theorem 3.3, an essentially unique posterior distribution of X given a sample y_0 of $\tilde{Y} = L(X) + \tilde{\varepsilon}$ can be represented as

$$\mu(U, y_0) = \frac{\int_U 1_K(y_0 - L(x)) \exp\left(\langle y_0, C_{\varepsilon}^{-1}L(x) \rangle - \frac{1}{2}\|L(x)\|_{H_{\mu_{\varepsilon}}}^2\right) d\mu_X(x)}{\int 1_K(y_0 - L(x)) \exp\left(\langle y_0, C_{\varepsilon}^{-1}L(x) \rangle - \frac{1}{2}\|L(x)\|_{H_{\mu_{\varepsilon}}}^2\right) d\mu_X(x)}$$

whenever the denominator is positive. We see that when we can exclude noise patterns, the posterior distribution will concentrate more on the true value x_0 (when L is injective). When $\mu_X(L^{-1}(\partial(\{y_0\} - K))) = 0$, the mapping $x \mapsto 1_K(y - L(x))$ is continuous on a set of full μ_X -measure and the convergence results are hence available.

For example, take $G = F = H^{-1}(a, b)$, where $-\infty < a < b < \infty$, and set $Lx = \sum_{i=1}^{\infty} c_i \langle x, e_i \rangle e_i$ for all $x \in L^2(a, b)$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of $L^2(a, b)$ and the constants $c_i > 0$ satisfy $\{(1+i)c_i\}_{i=1}^{\infty} \in \ell^2$. Then $L : L^2(a, b) \rightarrow H^1(a, b)$ is continuous. Set $X = \sum_{i=1}^{\infty} X_i e_i$ and $\varepsilon = \sum_{i=1}^{\infty} \varepsilon_i e_i$, where ε_i and X_i , $i \in \mathbf{N}$, are independent standard normal random variables. Set

$$K = \{y \in G : |\sum_{i=1}^k \langle y, e_i \rangle| \leq C\}.$$

Then $\mu_{\varepsilon}(K) > 0$ and

$$\begin{aligned} \mu_X(L^{-1}(\partial(\{y_0\} - K))) &= \mu_X(L^{-1}(\{y \in G : |\sum_{i=1}^k \langle y, e_i \rangle + \langle y_0, e_i \rangle| = C\})) \\ &= \mu_X(\{y \in L^2(a, b) : |\sum_{i=1}^k c_i (\langle y, e_i \rangle + \langle y_0, e_i \rangle)| = C\}) \\ &= \prod_{i=1}^k P(|Z| = C) = 0, \end{aligned}$$

where $Z = \sum_{i=1}^k c_i (X_i + \langle y_0, e_i \rangle)$ is a Gaussian random variable.

The partial uniqueness with respect to the topology of G remains an open question.

Another example arises from the Girsanov formula. We equip $G = C([0, T])$, where $T > 0$, with the usual supremum norm. The space G is then complete separable Banach space and its dual space G' is the space of Radon measures on $[0, T]$. We assume that the observation is of the form $Y_t = L(X)_t + \tilde{\varepsilon}_t$ for $0 \leq t \leq T$, where F -valued X and $C([0, T])$ -valued $\tilde{\varepsilon}$ are statistically independent and $L : F \rightarrow C([0, T])$ is a continuous mapping. More precisely, we assume a stronger condition that $L : F \rightarrow C_0^2(0, T)$ is continuous.

Suppose that the noise $\tilde{\varepsilon} \in G$ is of the form

$$\tilde{\varepsilon}_t = \varepsilon_t + \int_0^t a(s; \varepsilon_s) ds$$

where ε_t is an ordinary Brownian motion on $[0, T]$ and $a : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$ is continuous. Note that $\tilde{\varepsilon}_t$ indeed is a $C([0, T])$ -valued random variable since the continuous functionals $\{\delta_t : t \in \mathbf{Q} \cap [0, T]\}$ separate the points in G and, therefore, also generate the σ -algebra of G .

It is well-known that the Cameron-Martin space of the Brownian motion on $[0, T]$ is the separable Hilbert space $\{f \in H^1(0, T) : f(0) = 0\}$ equipped with the norm $\|f'\|_{L^2}$, the covariance operator C_{ε} has kernel $\min(t, s)$ and $C_{\varepsilon}^{-1} = \frac{d^2}{dt^2}$ on $\{f \in H^2(0, T) : f(0) = 0, f'(T) = 0\}$ (see [12]). By the Cameron-Martin theorem

$$\frac{d\mu_{\varepsilon+L(x)}}{d\mu_{\varepsilon}}(y) = \exp \left(\int_0^T y_s \frac{d^2 L(x)_s}{ds^2} ds - \frac{1}{2} \left\| \frac{dL(x)_s}{ds} \right\|_{L^2(0, T)}^2 \right).$$

The Girsanov formula

$$\frac{d\mu_{\tilde{\varepsilon}}}{d\mu_{\varepsilon}}(y) = \exp \left(\int_0^T a(s, y_s) dy_s - \frac{1}{2} \int_0^T |a(s, y_s)|^2 ds \right),$$

where the first integral is a sample of the corresponding stochastic integral, holds when the Novikov's condition

$$(24) \quad \mathbf{E} \left[\exp \left(\int_0^T |a(s; \varepsilon_s)|^2 ds \right) \right] < \infty$$

is satisfied (see [111]). For example, if $|a(s, x)| \leq C(1 + |x|)$ for some $C > 0$, then (24) holds since

$$\mathbf{E}[e^{C \int_0^T \varepsilon_s^2 ds}] \leq e^{\frac{C^2}{4\alpha}} \mathbf{E}[e^{a\|\varepsilon\|_{L^2(0,T)}^2}] < \infty$$

by the Fernique theorem (see Corollary 2.8.6 in [12]). For instance, take $a(s, x) = \frac{2x}{1+x^2}$. By the Itô formula, we see that the mapping

$$\varepsilon \mapsto \int_0^T a(\varepsilon_s) d\varepsilon_s = \ln(1 + \varepsilon_T^2) - \int_0^T \frac{1 - \varepsilon_s^2}{(1 + \varepsilon_s^2)^2} ds$$

extends to a continuous functional on $C([0, T])$. Thus $y \mapsto \frac{d\mu_{\tilde{\varepsilon}}}{d\mu_{\varepsilon}}(y)$ has a continuous version

$$\frac{d\mu_{\tilde{\varepsilon}}}{d\mu_{\varepsilon}}(y) = (1 + y_T^2) \exp \left(- \int_0^T \frac{1 - y_s^2}{(1 + y_s^2)^2} ds - \frac{1}{2} \int_0^T \left| \frac{2y_s}{1 + y_s^2} \right|^2 ds \right)$$

on $C([0, T])$. As in (22), we obtain an explicit solution

$$\mu(U, y) = \frac{\int_U \frac{d\mu_{\tilde{\varepsilon}}}{d\mu_{\varepsilon}}(y - L(x)) \frac{d\mu_{\varepsilon+L(x)}}{d\mu_{\varepsilon}}(y) d\mu_X(x)}{\int \frac{d\mu_{\tilde{\varepsilon}}}{d\mu_{\varepsilon}}(y - L(x)) \frac{d\mu_{\varepsilon+L(x)}}{d\mu_{\varepsilon}}(y) d\mu_X(x)}$$

of the statistical inverse problem of estimating the distribution of X given the observation $Y_t = L(X)_t + \varepsilon_t + \int_0^t a(s, \varepsilon_s) ds$ on $[0, T]$. The posterior convergence results are available for approximated prior distribution.

In general, any G -valued random variable $\tilde{\varepsilon}$ whose image measure is absolutely continuous with respect to a zero mean Gaussian measure μ_{ε} satisfies

$$\tilde{\varepsilon} = \varepsilon + T(\varepsilon)$$

in distribution for some mapping $T : G \rightarrow H_{\mu_{\varepsilon}}$ (see Corollary 4.2 in [14]).

5.4. Spherically invariant noise. Let F and G be locally convex Souslin topological vector spaces. We say that ε is a *spherically invariant G -valued random variable* if $\varepsilon = \gamma Z$, where Z is a zero-mean Gaussian G -valued random variable whose Cameron-Martin space is infinite-dimensional, and Z is statistically independent from a non-negative real-valued random variable γ whose distribution has no atom at zero.

The expression "spherically invariant random process (SIRP)" is used in the engineering literature [158] while the more descriptive but little used expression " H_{μ_Z} -spherically symmetric measure" appears in the mathematical literature (see Definition 7.4.1 in [12]). The latter has emphasis on the fact that the measure is only invariant with respect to orthogonal operators on H_{μ_Z} (see Theorem 7.4.2 in [12]).

In order to study the posterior measure of X given $Y = L(X) + \gamma Z$, we apply an averaging principle together with the following lemma.

Lemma 5.1. *Let F and G be locally convex Souslin topological vector spaces. Let Z be a zero-mean Gaussian G -valued random variable whose Cameron-Martin space is infinite-dimensional. Let X be an F -valued random variable, and let γ be a non-negative random variable whose distribution has no atom at zero. Suppose that γ , X and Z are statistically independent.*

Let $L : F \rightarrow G$ be a continuous mapping such that $L(F) \subset H_{\mu_Z}$, where H_{μ_Z} is the Cameron-Martin space of μ_Z . Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis of H_{μ_Z} such that $C_Z^{-1}e_i \in G'$, where C_Z is the covariance operator of Z . Set $Y = L(X) + \gamma Z$.

For any $f \in L^1(\mu_{(Y,\gamma)})$, the conditional expectation

$$\mathbf{E}[f(Y, \gamma) | \sigma(Y)](\omega) = f(Y(\omega), \gamma_{Y(\omega)})$$

for P -almost every $\omega \in \Omega$, where $y \mapsto \gamma_y$ is a \mathcal{G} -measurable function on G that satisfies

$$(25) \quad \gamma_y = \left(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \langle y, C_Z^{-1}e_i \rangle^2 \right)^{\frac{1}{2}}$$

whenever a finite limit exists and $\gamma_y = 0$ otherwise.

Proof. The mapping $y \mapsto \gamma_y$ is indeed measurable since the set

$$N = \{y \in G : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \langle y, C_Z^{-1}e_i \rangle^2 \neq \beta\}.$$

is a Borel set (see Lemma 2.1.7 in [13]). We show in a moment that $\gamma = \gamma_Y$ P -almost surely. Then the conditional expectations of $f(Y, \gamma)$ and $f(Y, \gamma_Y)$ coincide since the two random variables coincide almost surely. In order to conclude the claim, we note that $\gamma_{Y(\omega)}$ is $Y^{-1}(\mathcal{G})$ -measurable as a combination of two measurable functions.

The random variables γ , X and Z are statistically independent, which implies that their image measure $\mu_{(\gamma, X, Z)}$ is a product measure on the product space $\mathbf{R}_+ \times F \times G$.

Since Z has a Gaussian distribution, the random variables $\langle Z, C_Z^{-1}e_i \rangle$ are statistically independent standard normal random variables. The same holds for the random variables $(t, x, z) \mapsto \langle z, C_Z^{-1}e_i \rangle$ on the measure space $(\mathbf{R}_+ \times F \times G, \mathcal{B}(\mathbf{R}_+ \times F \times G), \mu_\gamma \otimes \mu_X \otimes \mu_Z)$. The random variable

$$(t, x, z) \mapsto L(x) + tz$$

has the following property. The law of large numbers implies that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \langle L(x) + tz, C_Z^{-1}e_i \rangle^2 = \lim_{n \rightarrow \infty} \frac{t^2}{n} \sum_{i=1}^n \langle z, C_Z^{-1}e_i \rangle^2 = t^2$$

for any $t \in \mathbf{R}_+$, $x \in F$, and μ_Z -a.e. $z \in G$. Since the image measure has the product structure, this also holds for $\mu_{(\gamma, X, Z)}$ -almost every (t, x, z) . Hence,

$$(\gamma, X, Z)^{-1} \left\{ (t, x, z) : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \langle L(x) + tz, C_Z^{-1}e_i \rangle^2 = t^2 \right\}$$

has full P -measure i.e.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \langle L(X) + \gamma Z, C_Z^{-1}e_i \rangle^2 = \gamma^2$$

P -almost surely. □

The averaging principle for the posterior distributions is given in the following lemma. Note, that also topological products of Souslin spaces are Souslin spaces.

Lemma 5.2. *Let the assumptions of the Lemma 5.1 hold.*

A solution $\mu(\cdot, y)$ of the statistical inverse problem of estimating the distribution of X given a sample y of $Y = L(X) + \gamma Z$ coincides μ_Y -almost surely with a Borel measurable solution $\tilde{\mu}(\cdot, (y, \gamma_y))$ of the statistical inverse problem of estimating the distribution of X given $(Y, \gamma) = (y, \gamma_y)$, where γ_y is defined by (25).

Proof. The σ -algebra $\sigma(Y)$ generated by $Y = L(X) + \gamma Z$ is a sub- σ -algebra of the σ -algebra $\sigma((Y, \gamma))$ generated by the $G \times \mathbf{R}_+$ -valued random variable $(Y, \gamma) = (\gamma Z + L(X), \gamma)$. By Lemma 5.1 and a property of conditional expectations, the solutions satisfy

$$\begin{aligned} \mu(U, Y) &= \mathbf{E}[1_U(X) | \sigma(Y)] = \mathbf{E}[\mathbf{E}[1_U(X) | \sigma(Y, \gamma)] | \sigma(Y)] \\ &= \mathbf{E}[\tilde{\mu}(U, (Y, \gamma)) | \sigma(Y)] = \tilde{\mu}(U, (Y, \gamma_Y)) \end{aligned}$$

almost surely for a fixed $U \in \mathcal{F}$. It is easy to see that $y \mapsto \tilde{\mu}(U, (y, \gamma_y))$ is Borel-measurable. By the Souslin property, it is enough to consider only countably many $U \in \mathcal{F}$ in order to identify the two measures. Hence, $(U, y) \mapsto \tilde{\mu}(U, (y, \gamma_y))$ is a solution of the statistical inverse problem of estimating the distribution of X given a sample y of $Y = L(X) + \gamma Z$. \square

Theorem 5.3. *Let the assumptions of the Lemma 5.1 hold. The essentially unique posterior distribution of X given a sample y of $Y = L(X) + \gamma Z$ has a version*

$$(26) \quad \mu(A, y) = \frac{\int_A \exp \left(\langle y, \gamma_y^{-2} C_Z^{-1} L(x) \rangle - \frac{1}{2\gamma_y^2} \|L(x)\|_{H_{\mu_Z}}^2 \right) d\mu_X(x)}{\int_F \exp \left(\langle y, \gamma_y^{-2} C_Z^{-1} L(x) \rangle - \frac{1}{2\gamma_y^2} \|L(x)\|_{H_{\mu_Z}}^2 \right) d\mu_X(x)},$$

for all $y \in G$ such that the limit

$$\gamma_y = \left(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \langle y, C_Z^{-1} e_i \rangle^2 \right)^{\frac{1}{2}}$$

exists and does not vanish.

Proof. Let us calculate the posterior distribution of X given (Y, γ) .

The conditional distribution of (Y, γ) given a sample x of X is $\mu_{(\gamma Z + L(x), \gamma)}(C \times B)$, where $C \in \mathcal{G}$ and $B \in \mathcal{B}(\mathbf{R}_+)$ by Lemma 3.2. Furthermore, the conditional distribution of $L(x) + \gamma Z$ given $\sigma((\gamma, X))$ is $\mu_{\gamma(\omega_0)Z + L(x)}$. Taking conditional expectations inside the integral gives

$$\begin{aligned} \mu_{(\gamma Z + L(x), \gamma)}(C \times B) &= P(\gamma Z + L(x) \in C \cap \gamma \in B) \\ &= \mathbf{E}[1_C(\gamma Z + L(x)) 1_B(\gamma)] \\ &= \mathbf{E}[\mathbf{E}[1_C(\gamma Z + L(x)) | \sigma(\gamma)] 1_B(\gamma)] \\ &= \int \mu_{aZ + L(x)}(C) 1_B(a) d\mu_\gamma(a). \end{aligned}$$

We may now use the absolute continuity of the translated measures $\mu_{aZ+L(x)}$ with respect to μ_{aZ} which follows from the Cameron-Martin theorem. We obtain

$$\begin{aligned}
\mu_{(\gamma Z+L(x), \gamma)}(C \times B) &= \int_B \int_C e^{\langle y, a^{-2} C_Z^{-1} L(x) \rangle - \frac{1}{2a^2} \|L(x)\|_{H_{\mu_Z}}^2} d\mu_{aZ}(y) d\mu_{\gamma}(a) \\
&= \int_{C \times B} e^{\langle y, a^{-2} C_Z^{-1} L(x) \rangle - \frac{1}{2a^2} \|L(x)\|_{H_{\mu_Z}}^2} d\mu_{aZ}(y) d\mu_{\gamma}(a) \\
&= \mathbf{E}[\mathbf{E}[1_{C \times B}(\gamma Z, \gamma) e^{\langle \gamma Z, \gamma^{-2} C_Z^{-1} L(x) \rangle - \frac{1}{2\gamma^2} \|L(x)\|_{H_{\mu_Z}}^2} | \sigma(\gamma)]] \\
&= \int_{C \times B} e^{\langle y, a^{-2} C_Z^{-1} L(x) \rangle - \frac{1}{2a^2} \|L(x)\|_{H_{\mu_Z}}^2} d\mu_{(\gamma Z, \gamma)}(y, a).
\end{aligned}$$

Hence, the Radon-Nikodym derivative of $\mu_{(\gamma Z+L(x), \gamma)}$ with respect to $\mu_{(\gamma Z, \gamma)}$ is

$$\frac{d\mu_{(\gamma Z+L(x), \gamma)}}{d\mu_{(\gamma Z, \gamma)}}(y, a) = \exp \left(\langle y, a^{-2} C_Z^{-1} L(x) \rangle - \frac{1}{2a^2} \|L(x)\|_{H_{\mu_Z}}^2 \right).$$

The posterior distribution of X given a sample (y, a) of (Y, γ) has a version

$$\mu(U, (y, a)) = \frac{\int_U \exp \left(\langle y, a^{-2} C_Z^{-1} L(x) \rangle - \frac{1}{2a^2} \|L(x)\|_{H_{\mu_Z}}^2 \right) d\mu_X(x)}{\int_F \exp \left(\langle y, a^{-2} C_Z^{-1} L(x) \rangle - \frac{1}{2a^2} \|L(x)\|_{H_{\mu_Z}}^2 \right) d\mu_X(x)}$$

for all $y \in G$ and $a \neq 0$. We obtain the required result by Lemma 5.2. \square

Posterior convergence holds under the same conditions as in the Gaussian case.

Remark 11. The posterior distribution (26) does not depend on the distribution of γ . Especially, γ does not necessarily have finite moments.

Remark 12. If the sample $y \in H_{\mu_Z}$, then the estimated random number $\gamma_y = 0$. Consequently, we can not apply Theorem 2.7 for the solution (26) on any measurable linear subspace of G of full μ_Z -measure, since it contains the Cameron-Martin space H_{μ_Z} . Besides the Lusin theorem, nothing seems to be known about the continuity of the measurable function $y \mapsto \gamma_y$. Even though the continuity of the posterior distribution as a function of observations remains an open question, we can anticipate from the form of the posterior distribution that the prior distribution will have a good regularizing effect on the corresponding ill-posed inverse problem.

Following [125], we call $\varepsilon = \gamma Z$ a *symmetric α -stable sub-Gaussian G -valued random variable* if $\gamma = \sqrt{\Gamma}$, where the non-negative random variable Γ satisfies

$$\mathbf{E}[e^{-t\Gamma}] = e^{-t^{\alpha/2}}, \quad t > 0,$$

for some $0 < \alpha < 2$, and Z is a zero mean G -valued Gaussian random variable.

For instance, α -stable random variables are used as approximative models for ambient noise. An example of ambient noise is the acoustic noise in oceans originating from e.g. shipping, rain fall, waves, animal activity, bubbles, cracking of ice and geological processes [66, 146]. It disturbs acoustic communication and active acoustic remote sensing in underwater environments [20, 92]. The finite-dimensional distributions of ambient noise are thought to originate from many disturbances occurring in natural environments: typically few strong and a large number of weak disturbances of different orders. The variances of individual disturbances are often such that Lindeberg's condition, which is a sufficient condition (and in some cases also necessary) for the applicability of the classical central limit theorem, does not hold [152]. A generalized central limit theorem states that a.s. converging sums

of independent random variables necessarily have stable distributions (see Definition 1.1.5 in [125]). Non-Gaussian stable distributions exhibit heavy tails, which explains why the Gaussian distributions are not the best ones for modeling ambient noise. Symmetric α -stable sub-Gaussian random variables are perhaps the most simple subclass of stable distributions.

Sub-Gaussian noise is encountered also in fMRI (functional magnetic resonance imaging), where it models physiological noise, e.g. disturbances originating from breathing and heartbeat [17].

Spherically symmetric noise models are used also as approximative models in high resolution radar imaging for describing the ground-clutter (i.e. unwanted echoes of the transmitted radar signal from the ground), and also sea-clutter (i.e. echoes from the surface of the sea) [22, 23, 24]. It should be noted that the modeling of radar clutter and underwater noise is not yet a mature field of science. Beside of spherically symmetric models also other models have been developed and better models are pursued after.

Noise is usually rougher than the signal by rule of thumb. In the above applications, it is not verified whether this holds for the noise ε and signals $L(x)$, where $x \in F$. For radar imaging this is not a critical point since the reflected signal acquires some regularity from the transmitted signal.

5.5. Subordinated noise. We consider another generalization of Gaussian noise that is similar to spherically symmetric noise.

Let B_t be a Brownian motion on \mathbf{R}_+ satisfying $B_0 = 0$ almost surely. Subordinated noise is here defined as a time-changed process

$$\varepsilon_t = B_{\alpha_t},$$

where α_t is a strictly increasing stochastic process that is statistically independent from the Brownian motion B_t . We assume that α_t has bi-Lipschitz-continuous sample paths and satisfies $\alpha_0 = 0$. For example, α_t can be an integral function of some statistically independent Gamma process starting from a non-zero value. Such a distribution of α can reflect inaccuracies that are believed to be present in the covariance operator of the noise ε .

Lemma 5.4. *The random function ε . on $[0, 1]$ is a $C([0, 1])$ -valued random variable.*

Proof. The sample paths of ε are continuous functions as compositions of continuous functions. Moreover, the space $C([0, 1])$ is a separable Fréchet space, which implies that its Borel σ -algebra is generated by the cylinder sets

$$A = \{f \in C([0, 1]) : f(t_i) \in U_i \forall i \in I\}$$

where $U_i \in \mathcal{B}(\mathbf{R})$, $I \subset \mathbf{N}$ are finite sets, and $\cup_{i=1}^{\infty} t_i$ is a dense subset of $[0, 1]$. (see Theorem A.3.7 in [12]). It is enough to check that the mapping

$$\omega \mapsto B_{\alpha_{t_i}}$$

is a random variable for any $t_i \in [0, 1]$. But this follows from the joint measurability of the Brownian motion from $[0, 1] \times \Omega$ into \mathbf{R} . \square

We take $G = C([0, 1])$, $\mathcal{G} = \mathcal{B}(C([0, 1]))$, and denote $\mu_{\varepsilon}(A) = P(\varepsilon \in A)$ for any Borel set $A \subset C([0, 1])$.

Lemma 5.5. *The Gaussian measure $\mu_{B_{\alpha(\omega)} + L(x)}$ that has mean $L(x)$ and the covariance operator with kernel $C_{\alpha(\omega)}(t, s) = \min(\alpha_t(\omega), \alpha_s(\omega))$ on $[0, 1] \times [0, 1]$ is a version of the conditional probability $V \mapsto \mathbf{E}[1_V(B_{\alpha} + L(x)) | \sigma(\alpha)](\omega)$ on \mathcal{G} .*

Proof. By defining $B_t = 1_{t \geq 0} B_t$ on \mathbf{R} , the Brownian motion extends to a $C(\mathbf{R})$ -valued random variable, where $C(\mathbf{R})$ is equipped with the Borel σ -algebra with respect to the locally convex topology given by the family of seminorms

$$\rho_i(f) = \sup_{t \in K_i} |f(t)|,$$

where $K_i = [-i, i]$ and $i \in \mathbf{N}$ (i.e. the topology of uniform convergence on compact sets). The space $C(\mathbf{R})$ is then a locally convex Souslin space, since its topology is metrizable by a complete metric

$$d(f_1, f_2) = \sum_{i=1}^{\infty} 2^{-i} \frac{\rho_i(f_1 - f_2)}{1 + \rho_i(f_1 - f_2)}$$

and the polynomials with rational coefficients form a dense set by the Stone-Weierstrass theorem. Moreover, α is a $C([0, 1])$ -valued random variable.

Recalling Lemma 3.2, we need to check that the composition mapping $(f, g) \mapsto f \circ g + L(x)$ is Borel measurable from $C(\mathbf{R}) \times C([0, 1])$ into $C([0, 1])$. Since point evaluations generate the Borel σ -algebra of $C([0, 1])$, it is enough to show that functionals $(f, g) \mapsto f \circ g(t) + L(x)_t$ are Borel measurable for a fixed $t \in [0, 1]$. We show that this function is actually continuous. Since the both spaces are metric spaces, it is enough to check the sequential continuity on the product space $C(\mathbf{R}) \times C([0, 1])$, which is metrizable.

Let $\lim_{i \rightarrow \infty} (f_i, g_i) = (f, g)$ in $C(\mathbf{R}) \times C([0, 1])$, which implies that $\lim_{i \rightarrow \infty} f_i = f$ and $\lim_{i \rightarrow \infty} g_i = g$ in corresponding spaces. Then $K = \{g_i(t) \in \mathbf{R} : i \in \mathbf{N}\}$ is compact for the fixed $t \in [0, 1]$ and

$$\begin{aligned} |f_i(g_i(t)) - f(g(t))| &= |f_i(g_i(t)) - f(g_i(t)) + f(g_i(t)) - f(g(t))| \\ &< \sup_{t \in K} |f_i(t) - f(t)| + |f(g_i(t)) - f(g(t))| \rightarrow 0 \end{aligned}$$

as $i \rightarrow \infty$ by the convergence of (f_i, g_i) and the continuity of f . \square

Theorem 5.6. *Let F be a locally convex Souslin topological vector space equipped with its Borel σ -algebra \mathcal{F} and let $L : F \rightarrow H^1([0, 1])$ be a continuous mapping that satisfies $L(x)|_{t=0} = 0$ for all $x \in F$. Let B_t be a Brownian motion on $[0, 1]$ starting from zero. Let α_t be a strictly increasing stochastic process that is statistically independent from the Brownian motion B_t and that has bi-Lipschitz continuous sample paths satisfying $\alpha(0) = 0$ almost surely. Let X be an F -valued random variable that is statistically independent from the Brownian motion B_t and the stochastic process α_t .*

The essentially unique solution of estimating the distribution of X given a sample path $y : [0, 1] \rightarrow \mathbf{R}$ of $Y_t = L(X)_t + B_{\alpha_t}$ has a version μ such that

$$\mu(U, y) = \frac{\int_U \exp \left(\langle y, C_{B[y]}^{-1} L(x) \rangle - \frac{1}{2} \|L(x)\|_{H_{\mu_{B[y]}}}^2 \right) d\mu_X(x)}{\int \exp \left(\langle f, C_{B[y]}^{-1} L(x) \rangle - \frac{1}{2} \|L(x)\|_{H_{\mu_{B[y]}}}^2 \right) d\mu_X(x)}$$

for any $U \in \mathcal{F}$ and for any $y \in C([0, 1])$ such that its quadratic variation $[y]$ satisfies $0 < [y]_t < \infty$ for all $t \in (0, 1]$.

Proof. Let g be some sample of α on $[0, 1]$. The mapping

$$T_g : f \mapsto f \circ g$$

is linear and measurable from $C(\mathbf{R}_+)$ to $C([0, 1])$. Hence, the Cameron-Martin space of $T_g B = B_g$ coincides with $T(H_0^1(\mathbf{R}_+))$ as a vector space (see Theorems 3.7.3 and 3.7.6 in [12]; choose $X = C(\mathbf{R}_+) \times C([0, 1])$ in order to generalize the claim to the present situation). Since g is bi-Lipschitz continuous, the mapping $f \circ g$ is in $H^1(0, 1)$ whenever $f \in H^1(g(0, 1))$ (e.g. Theorem 2.2.2 in [165]), and the mapping is actually onto the subspace $H = \{f \in H^1(0, 1) : f(0) = 0\}$. Especially, the vector $L(x) \in H_{\mu_{B_g}}$ by the assumption, so by Lemma 5.5

$$\frac{d\mu_{(B_\alpha + L(x), \alpha)}}{d\mu_{(B_\alpha, \alpha)}}(z, \alpha) = \exp \left(\langle z, C_\alpha^{-1} L(x) \rangle - \frac{1}{2} \|L(x)\|_{H_{\mu_{B_\alpha}}}^2 \right)$$

as in Theorem 5.3. It is well-known that for continuous time-changes α_t the quadratic variation of B_{α_t} coincides with α_t (see Chapter 5: Proposition 1.5 in [121]). Therefore, α_t is a measurable function of the sample paths of B_{α_t} (the quadratic variation is obtained by taking a limit in probability and we need to pick up a subsequence in order to get the a.s. convergence). Since $L(X) \in H^1(0, 1)$, it has finite variation, which implies that its quadratic variation vanishes. Also μ_Y -almost every sample path of $L(X)_t + B_{\alpha_t}$ has α_t as its quadratic variation. We obtain the claim similarly as in Lemma 5.2. \square

Posterior convergence holds similarly as in the Gaussian case. The assumptions that guarantee the continuity of the solution are not known.

5.6. Decomposable additive noise. Let F and G be locally convex Souslin topological vector spaces. We say that G -valued random noise ε is *decomposable* if it is of the form

$$\varepsilon = \sum_{i=1}^{\infty} \varepsilon_i f_i,$$

where ε_i are independent random variables with a.e. positive probability density functions ρ_i with respect to the Lebesgue measure and $f_i \in G$ are some non-zero vectors.

Remark 13. If ε_i are random variables and $\varepsilon := \sum_{i=1}^{\infty} \varepsilon_i f_i$ a.s. for some vectors $f_i \in G$, then ε is a G -valued random variable. Indeed, since G is a Souslin topological vector space, the mapping $\mathbf{R} \times G \ni (a, f) \mapsto af =: T(a, f)$ is continuous, therefore also $\mathcal{B}(R \times G) = \mathcal{B}(R) \otimes \mathcal{G}$ measurable. The composition of the measurable mapping $(\omega, f) \mapsto (\varepsilon_i(\omega), f)$ with T gives a G -valued random variable $T(\varepsilon_i, f) = \varepsilon_i f$. Also the sum of two G -valued random variables is a G -valued random variable and limits of locally convex Souslin space-valued random variables are random variables (since the cylinder sets generate the Borel σ -algebra by Theorem 6.8.9 in [13]).

If all possible signals $L(x)$, $x \in F$ are sparse in the sense that they belong to the linear span of $\{f_i : i \in \mathbf{N}\}$ and the noise ε is decomposable, then the measures $\mu_{\varepsilon + L(x)}$ are absolutely continuous with respect to μ_ε [126].

Moreover, if $\{f_i\}_{i=1}^{\infty}$ is a basis of the closed subspace $\overline{\text{span}(\{f_i : i \in N\})}$, the proof in [126] gives, with minor additional work, an explicit formula for the Radon-Nikodym density. For simplicity, we take $G = \overline{\text{span}(\{f_i : i \in N\})}$.

Theorem 5.7. *Let G be a locally convex Souslin topological vector space equipped with the Borel σ -algebra \mathcal{G} and a basis $\{f_i\}_{i=1}^{\infty}$ such that the unique coefficients y_i in $y = \sum_{i=1}^{\infty} y_i f_i$ depend measurably on $y \in G$. Let a G -valued random variable ε be of the form $\varepsilon = \sum_{i=1}^{\infty} \varepsilon_i f_i$, where the random variables ε_i are statistically*

independent and have probability density functions ρ_i that are a.e. positive. If $L_n(x) = \sum_{i=1}^n a_i(x)f_{k_i}$, then

$$(27) \quad \frac{d\mu_{\varepsilon+L_n(x)}}{d\mu_{\varepsilon}}(y) = \frac{\prod_{i=1}^n \rho_{k_i}(y_{k_i} - a_i(x))}{\prod_{i=1}^n \rho_{k_i}(y_{k_i})}$$

for almost every $y = \sum_{i=1}^{\infty} y_i f_i$.

Proof. Let $A \in \mathcal{G}$. By possibly rearranging finitely many vectors, we may suppose that $L_n(x) = \sum_{i=1}^n a_i f_i$. We consider the probability

$$\mu_{\varepsilon+L_n(x)}(A) = P(\varepsilon + L_n(x) \in A) = \mathbf{E} \left[1_A \left(\sum_{i=1}^{\infty} \varepsilon_i f_i + \sum_{i=1}^n a_i f_i \right) \right].$$

Denote $Z = \sum_{i=n+1}^{\infty} \varepsilon_i f_i$. Following [126], we calculate the conditional expectation of $1_A(\varepsilon + L(X))$ given $\sum_{i=1}^n (\varepsilon_i + a_i) f_i$ and, by Lemma 3.2, obtain with straightforward calculations

$$\begin{aligned} \mathbf{E} \left[1_A \left(\sum_{i=1}^{\infty} \varepsilon_i f_i + \sum_{i=1}^n a_i f_i \right) \right] &= \mathbf{E} \left[\mathbf{E} \left[1_A \left(Z + \sum_{i=1}^n (\varepsilon_i + a_i) f_i \right) \mid \sum_{i=1}^n (\varepsilon_i + a_i) f_i \right] \right] \\ &= \int \mu_{Z + \sum_{i=1}^n (\varepsilon_i(\omega) + a_i) f_i}(A) dP(\omega) \\ &= \int_{\mathbf{R}^n} \mu_{Z + \sum_{i=1}^n (y_i + a_i) f_i}(A) \left(\prod_{i=1}^n \rho_i(y_i) \right) dy_1 \cdots dy_n \\ &= \int_{\mathbf{R}^n} \mu_{Z + \sum_{i=1}^n y_i f_i}(A) \left(\prod_{i=1}^n \rho_i(y_i - a_i) \right) dy_1 \cdots dy_n. \end{aligned}$$

At this point, the proof differs from [126]. Namely, we multiply and divide with the positive densities of ε_i , and obtain

$$\begin{aligned} \mathbf{E} \left[1_A \left(\sum_{i=1}^{\infty} \varepsilon_i f_i + \sum_{i=1}^n a_i f_i \right) \right] &= \int \mu_{Z + \sum_{i=1}^n y_i f_i}(A) \prod_{i=1}^n \frac{\rho_i(y_i - a_i)}{\rho_i(y_i)} \rho_i(y_i) dy_1 \cdots dy_n \\ &= \int \mu_{Z + \sum_{i=1}^n \varepsilon_i(\omega) f_i}(A) \prod_{i=1}^n \frac{\rho_i(\varepsilon_i(\omega) - a_i)}{\rho_i(\varepsilon_i(\omega))} dP(\omega) \\ &= \mathbf{E} \left[\mathbf{E} \left[1_A \left(\sum_{i=1}^{\infty} \varepsilon_i f_i \right) \mid \sum_{i=1}^n \varepsilon_i f_i \right] \prod_{i=1}^n \frac{\rho_i(\varepsilon_i - a_i)}{\rho_i(\varepsilon_i)} \right] \\ &= \mathbf{E} \left[1_A(\varepsilon) \prod_{i=1}^n \frac{\rho_i(\varepsilon_i - a_i)}{\rho_i(\varepsilon_i)} \right]. \end{aligned}$$

Since the unique coefficients ε_i depend measurably on ε , we may write

$$\begin{aligned} \mu_{\varepsilon+L_n(x)}(A) &= \mathbf{E} \left[1_A(\varepsilon) \prod_{i=1}^n \frac{\rho_i(\varepsilon_i - a_i)}{\rho_i(\varepsilon_i)} \right] \\ &= \int 1_A(y) \prod_{i=1}^n \frac{\rho_i(y_i - a_i)}{\rho_i(y_i)} d\mu_{\varepsilon}(y). \end{aligned}$$

□

The above theorem verifies the intuitive picture that for sparse signals we may as well study the posterior of X given the finite-dimensional data

$$Y_n = \sum_{i=1}^n (a_i(X) f_{k_i} + \varepsilon_i f_{k_i}).$$

The following theorem gives a significant enlargement of applicable noise models in statistical inverse problems.

Corollary 2 (Generalized Cameron-Martin formula). *Let G be a locally convex Souslin topological vector space equipped with the Borel σ -algebra \mathcal{G} and a basis $\{f_i\}_{i=1}^\infty$ such that the unique coefficients y_i in $y = \sum_{i=1}^\infty y_i f_i$ depend measurably on $y \in G$. Let a G -valued random variable ε be of the form $\varepsilon = \sum_{i=1}^\infty \varepsilon_i f_i$, where the random variables ε_i are statistically independent and have probability density functions ρ_i that are a.e. positive. If $L(x) = \sum_{i=1}^\infty a_i(x) f_i$ for all $x \in F$, and densities*

$$\frac{d\mu_{\varepsilon+L_n(x)}}{d\mu_\varepsilon}(y) = \frac{\prod_{i=1}^n \rho_i(y_i - a_i(x))}{\prod_{i=1}^n \rho_i(y_i)}$$

are uniformly integrable with respect to μ_ε and convergent μ_ε -almost everywhere, then

$$\frac{d\mu_{\varepsilon+L(x)}}{d\mu_\varepsilon}(y) = \prod_{i=1}^\infty \frac{\rho_i(y_i - a_i(x))}{\rho_i(y_i)}$$

for μ_ε -almost every $y = \sum_{i=1}^\infty y_i f_i$.

Proof. See Proposition 9.9.10 in [13], which says that if $\lim_n T_n = T$, where T_n and T are measurable mappings on a completely regular space, and the distributions of all T_n have uniformly integrable Radon-Nikodym densities ρ_n with respect to the same Radon probability measure ν , then the distribution of T has Radon-Nikodym density ρ with respect to the same Radon probability measure as well, and ρ is the limit of ρ_n in the weak topology of $L^1(\nu)$. This result is especially applicable to the random variables $T_n(x, z) = L_n(x) + z$ and $T(x, z) = L(x) + z$ on $(F \times G, \mathcal{F} \otimes \mathcal{G}, \mu_X \otimes \mu_\varepsilon)$ and the measure $\nu = \mu_\varepsilon$. The integrals of the densities over any Borel set converge. By Theorem 4.5.6 and Corollary 4.5.7 in [13] the weak limit coincides with the almost sure limit. \square

In Corollary 2, the Radon-Nikodym density $\frac{d\mu_{\varepsilon+L(x)}}{d\mu_\varepsilon}(y)$ has a form similar to Radon-Nikodym densities appearing in the Kakutani dichotomy theorem, which addresses the equivalence and singularity of infinite product measures on \mathbf{R}^∞ [77]. Also Umemura [145] has given conditions for the absolute continuity of measures on abstract spaces when the corresponding finite-dimensional distributions are absolutely continuous. In our case, Umemura's conditions ask $\left(\frac{d\mu_{\sum_{i=1}^n (\varepsilon_i + L(x)_i)}}{d\mu_{\sum_{i=1}^n \varepsilon_i}}(y) \right)^{\frac{1}{2}}$ to be a Cauchy sequence in $L^2(\mu_\varepsilon)$. We feel that the uniform integrability of the Radon-Nikodym densities is easier to validate than Umemura's conditions.

Corollary 3. *Let the assumptions of Corollary 2 hold. The essentially unique posterior distribution of X given a sample y of $Y = L(X) + \varepsilon$ has a version $\mu(\cdot, y)$ such that*

$$\mu(U, y) = \frac{\int_U \prod_{i=1}^\infty \frac{\rho_i(y_i - L(x)_i)}{\rho_i(y_i)} d\mu_X(x)}{\int \prod_{i=1}^\infty \frac{\rho_i(y_i - L(x)_i)}{\rho_i(y_i)} d\mu_X(x)}$$

whenever $0 < \int \prod_{i=1}^\infty \frac{\rho_i(y_i - L(x)_i)}{\rho_i(y_i)} d\mu_X(x) < \infty$.

Remark 14. Signals in non-Gaussian noise may appear in model approximations. Let the true model be

$$Y = L(X) + \varepsilon$$

where $\varepsilon = \sum_{i=1}^{\infty} \varepsilon_i f_i$ and all ε_i are statistically independent. When the model L is numerically very complicated, the common practice is to replace L with some simpler approximation L_n . For example, L_n may have the form $L_n(X) = \sum_{i=1}^n a_i(X) f_i$. Though the true model L and the approximated model L_n are known, the model error $\tilde{\varepsilon} = L(X) - L_n(X)$ is sometimes replaced with a G -valued random variable ε' that has the same distribution as $\tilde{\varepsilon}$ but is statistically independent from X [140]. We note that the observation model is then

$$Y = L_n(X) + \varepsilon' + \varepsilon,$$

where ε represents the uncertainties in the forward model L_n . Beside of physical noise, the distribution of the noise may represent our prior beliefs about the uncertainties in the forward model, which do not necessarily have Gaussian distributions.

5.7. Periodic signals in decomposable Laplace noise. In this section, we study an example case of the generalized Cameron-Martin formula for a non-Gaussian noise distribution. A similar distribution has been constructed before by Shimomura [131] who gave conditions under which certain translates of the distribution were equivalent to the original distribution. However, we use the methods of Section 5.6.

One class of inverse problems that involves periodic signals are the inverse scattering problems – the far-field pattern of the scattered wave in the 2D fixed energy inverse acoustic or potential scattering problem is a function on the torus. The measured far-field pattern is possibly contaminated by instrumental noise, far-fields of other unknown incoming fields, contributions from other scatterers, and the near-field and plane wave approximation errors. Although the random model below is oversimplified to fully cover this case, it shows how periodicity can be utilized in Bayesian inverse problems.

Suppose that $L(x) \in C^\alpha(S^1)$ for all $x \in F$ and some $\alpha > 1$. Then the Fourier coefficients

$$\widehat{L(x)}_k = \frac{1}{2\pi} \int_0^{2\pi} L(x; t) e^{-ikt} dt$$

are ℓ_1 -summable and the corresponding Fourier series converges to the limit

$$L(x; t) = \sum_{k=-\infty}^{\infty} \widehat{L(x)}_k e^{ikt}$$

in $C(S^1)$ (equipped with the usual supremum norm).

Let ε_k be mutually statistically independent random variables whose probability density functions with respect to the Lebesgue measure are

$$\rho_k(t) = \frac{1}{2b} e^{-|t|/b}$$

for all $k \in \mathbf{Z}$ and some common $b > 0$ i.e. they are zero mean Laplace random variables. The relation of the normal distribution to the Laplace distribution is that a conditionally normal random variable $\tilde{\varepsilon}_k | \sigma \sim N(0, \sigma^2)$ with a Rayleigh distributed variance has a Laplace distribution. In statistical inverse problems, one interpretation of the Laplace distribution is that we do not know the error variance exactly and are lead to describe our lack of knowledge in the form of a probability distribution.

Lemma 5.8. *The random sum*

$$\varepsilon = \sum_{k=-\infty}^{\infty} \varepsilon_k e^{ikt}$$

converges in $H^{-1}(S^1)$ and

$$\langle \varepsilon, e^{-ikt} \rangle_{H^{-1}(S^1), H^1(S^1)} = \varepsilon_k$$

Proof. The space $H^{-1}(S^1)$ is a Hilbert space, and the random variables ε_k have zero mean so it suffices to prove that $\sum_k E[\|\varepsilon_k e^{ikt}\|_{H^{-1}(S^1)}^2] < \infty$ (see Theorem 2 in Chapter 3.2 in [73]). The sequence $\{e^{ikt}\}_{k=-\infty}^{\infty}$ forms an orthonormal basis of $L^2(S^1)$. The imbedding of $L^2(S^1)$ into $H^{-1}(S^1)$ is Hilbert-Schmidt by Maurin's theorem, which implies that

$$\sum_k E[\|e^{ikt}\|_{H^{-1}(S^1)}^2] < \infty$$

and therefore

$$\mathbf{E} \sum_{k=-\infty}^{\infty} |\varepsilon_k|^2 \|e^{ikt}\|_{H^{-1}(S^1)}^2 \leq C \sum_k \|e^{ikt}\|_{H^{-1}}^2 < \infty.$$

Here we used the fact that the variance of the Laplace random variable ε_i is $2b^2$. \square

Next, we quickly check that ε is a non-Gaussian random variable. The characteristic function of the random variable ε is

$$\widehat{\mu}_\varepsilon(\phi) = \mathbf{E}[e^{i\langle \varepsilon, \phi \rangle_{H^{-1}(S^1), H^1(S^1)}}] = \prod_{k=-\infty}^{\infty} \frac{1}{1 + b^2(\widehat{\phi}_{-k})^2},$$

where $\widehat{\phi}_k$ is the Fourier coefficient $\frac{1}{2\pi} \int_0^{2\pi} \phi(t) e^{-ikt} dt$ of $\phi \in H^1(S^1)$. Note, that when $\phi_j \rightarrow \psi \in L^2(S^1)$ as $j \rightarrow \infty$, then

$$\lim_{j \rightarrow \infty} \widehat{\mu}_\varepsilon(\phi_j) = \prod_{k=-\infty}^{\infty} \frac{1}{1 + b^2(\widehat{\psi}_k)^2} =: \widehat{\mu}_\varepsilon(\psi)$$

i.e. their distributions converge weakly. Especially, when $\widehat{\psi}_k = \frac{1}{\pi(|k| - \frac{1}{2})}$, $k \neq 0$ and $\widehat{\psi}_0 = 0$, then

$$\widehat{\mu}_\varepsilon(t\psi) = \frac{1}{\cosh^2(bt)}.$$

Since the weak limits of zero mean Gaussian distributions are always zero mean Gaussian distributions, this shows that ε is indeed a non-Gaussian random variable.

We wish to study the statistical inverse problem of estimating the probability distribution of X when a sample of

$$Y = L(X) + \varepsilon$$

is known. This means that the inexact observations of the Fourier coefficients of $L(X)$ are assumed to be similarly inaccurate and some components are allowed to have high inaccuracies. Since Laplace distribution has heavier tails than the Gaussian distribution, it protects against outliers better than the normal distribution.

Consider first finite sums

$$L_n(x; t) = \sum_{|k| \leq n} \widehat{L(x)}_k e^{ikt}.$$

By Lemma 5.8 and Theorem 5.7, the Radon-Nikodym derivative of the translated measure $\mu_{\varepsilon+L_n(x)}$ with respect to the measure μ_ε on $H^{-1}(S^1)$ is

$$\begin{aligned} \frac{d\mu_{\varepsilon+L_n(x)}}{d\mu_\varepsilon}(y) &= \frac{\prod_{k=-n}^n \rho_k(\widehat{y}_k - \widehat{L(x)}_k)}{\prod_{k=-n}^n \rho_k(\widehat{y}_k)} \\ &= \prod_{k=-n}^n e^{-b^{-1}|\widehat{y}_k - \widehat{L(x)}_k| + b^{-1}|\widehat{y}_k|}. \end{aligned}$$

By the triangle inequality, we obtain that

$$||\widehat{y}_k| - |\widehat{y}_k - \widehat{L(x)}_k|| \leq |\widehat{L(x)}_k|,$$

which are summable. Therefore, the limit

$$\sum_{k=-\infty}^{\infty} (|\widehat{y}_k| - |\widehat{y}_k - \widehat{L(x)}_k|)$$

exists.

Random variables $\varepsilon + L_n(x)$ converge almost surely to $\varepsilon + L(x)$. Therefore, corresponding measures converge weakly i.e. for all Borel sets A whose boundary is $\mu_{\varepsilon+L(x)}$ -zero measurable it holds that

$$\begin{aligned} \mu_{\varepsilon+L(x)}(A) &= \lim_{n \rightarrow \infty} \mu_{\varepsilon+L_n(x)}(A) \\ &= \lim_{n \rightarrow \infty} \int_A e^{b^{-1} \sum_{k=1}^n (|\widehat{y}_k| - |\widehat{y}_k - \widehat{L(x)}_k|)} d\mu_\varepsilon(y) \\ &= \int_A e^{b^{-1} \sum_{k=-\infty}^{\infty} (|\widehat{y}_k| - |\widehat{y}_k - \widehat{L(x)}_k|)} d\mu_\varepsilon(y) \end{aligned}$$

by the Lebesgue dominated convergence theorem. The exponential function is the Radon-Nikodym derivative $\frac{d\mu_{\varepsilon+L(x)}}{d\mu_\varepsilon}(y)$.

We have shown the following theorem.

Theorem 5.9. *Let F be a locally convex Souslin topological vector space equipped with its Borel σ -algebra. Let X be an F -valued random variable and let $L : F \rightarrow C^\alpha(S^1)$ be a continuous mapping for some $\alpha > 1$. Let*

$$\varepsilon = \sum_k \varepsilon_k e^{ikt}$$

be a $H^{-1}(S^1)$ -valued random variable such that all ε_k , where $k \in \mathbf{Z}$, are mutually statistically independent random variables with probability density functions

$$\rho_k(t) = \frac{1}{2b} e^{-|t|/b}$$

with respect to the Lebesgue measure for some $b > 0$.

The solution of the statistical inverse problem of estimating the distribution of X given a sample $y \in H^{-1}(S^1)$ of $Y = L(X) + \varepsilon$ is essentially unique and has a version μ such that

$$\mu(U, y) = \frac{\int_U e^{b^{-1} \sum_{k=-\infty}^{\infty} (|\widehat{y}_k| - |\widehat{y}_k - \widehat{L(x)}_k|)} d\mu_X(x)}{\int_U e^{b^{-1} \sum_{k=-\infty}^{\infty} (|\widehat{y}_k| - |\widehat{y}_k - \widehat{L(x)}_k|)} d\mu_X(x)}$$

for all $U \in \mathcal{F}$ and for all $y \in H^{-1}(S^1)$ such that the denominator is finite and non-zero.

Remark 15. The convergence of posterior distributions holds for example if for all $n \in \mathbf{N}$ it holds that $\mathbf{E}[e^{b^{-1}\|\widehat{L}(X_n)\|_{\ell^1}}] \leq C$ and $\mathbf{E}[e^{b^{-1}\|\widehat{L}(X)\|_{\ell^1}}] \leq C$ for some $C > 0$. Under the same conditions, the posterior distributions depend continuously on the observations. Indeed, the posterior distributions are sequentially continuous by the Lebesgue dominated convergence theorem and the continuity of $y \mapsto \hat{y}_k$, and sequentially continuous functions on Hilbert spaces are continuous. The topological support of μ_Y coincides with the topological support of μ_ε by Lemma 3.5. The topological support of μ_ε coincides with the closure of the linear span of $\{e^{ikt}\}_k$ in $H^{-1}(S^1)$, that is, $H^{-1}(S^1)$ (see [126]). By Theorem 2.7, μ is the only posterior distribution that depends continuously on the observations.

6. EXAMPLES OF PRIOR APPROXIMATIONS

We present some methods for approximating the unknown. We take the unknown X always to be statistically independent from the noise ε . Especially, when the Radon-Nikodym density $\rho(x, z) = \frac{d\mu_{\varepsilon+L(x)}}{d\nu}(z)$ is bounded and continuous, we do not need to ask anything special on X or its converging approximations in order to obtain posterior convergence.

6.1. Random series. As discussed in Section 1.4, random series and wavelet expansions are important devices in defining infinite-dimensional prior models. If a random variable X in a locally convex Sauslin topological vector space F can be expressed as an almost surely converging series $X = \sum_{i=1}^{\infty} Z_i \phi_i$, where $\phi_i \in F$ and Z_i are ordinary random variables, we obtain immediately finite-dimensional approximations $X_n := \sum_{i=1}^n Z_i \phi_i$ by truncation. The almost sure convergence of random variables X_n to X implies (by the Lebesgue dominated convergence theorem) that the distributions μ_{X_n} converge weakly to μ_X . We return to this topic in connection with the linear discretizations of X in Section 6.6.

6.2. Gaussian priors.

6.2.1. Gaussian random series. All Gaussian F -valued random variables can be expressed with random series expansions [12]. A typical example is the Karhunen-Loève expansion of a zero mean $L^2(a, b)$ -valued Gaussian random variables X , where ϕ_i are chosen to be normed eigenfunctions of the covariance operator

$$Cf(t) = \int_a^b \mathbf{E}[X_t X_s] f(s) ds$$

on $L^2(a, b)$. Then X can be expressed as $X = \sum_{i=1}^{\infty} Z_i e_i$, where the standard normal random variables Z_i are statistically independent, and the truncated series X_n gives an almost surely converging approximation of X . The almost sure convergence of X_n to X implies the weak convergence of μ_{X_n} to μ_X .

6.2.2. Converging covariances. A well-known sufficient condition for the weak convergence of probability measures m_n to a probability measure m on a locally convex Sauslin topological vector space F is that

- (i) the measures m_n are uniformly tight i.e. for every $\varepsilon > 0$ there exist a compact set $K_\varepsilon \subset F$ such that $\sup_n m_n(K_\varepsilon^C) < \varepsilon$.
- (ii) Characteristic functionals $\hat{m}_n(\phi)$ converge to $\hat{m}(\phi)$ for every $\phi \in F'$.

According to Prohorov's theorem, (i) implies that each subsequence of m_n has a weakly convergent subsequence (see Theorem 8.6.7 in [13]). Part (ii) identifies the limits of different subsequences. For some spaces, (i) can be deduced from (ii) and the known properties of \hat{m} (see [147]). These spaces include \mathbf{R}^n and the distribution spaces $\mathcal{S}'(\mathbf{R}^d)$ and $\mathcal{D}'(U)$, $U \subset \mathbf{R}^d$ open.

Let μ_n and μ be Gaussian zero mean measures on the distribution space $\mathcal{D}'(U)$, where $U \subset \mathbf{R}^n$ is open. If the covariance operators C_n of μ_n converge weakly to the covariance operator C of μ , i.e.

$$\lim_{n \rightarrow \infty} \langle C_n \phi, \psi \rangle = \langle C \phi, \psi \rangle$$

for every $\phi, \psi \in \mathcal{D}(U)$, then (i) holds. Moreover, the characteristic functionals $\hat{\mu}_n$ are then equicontinuous at zero, which is sufficient for the uniform tightness of the sequence μ_n by Corollary 7.13.10 in [13].

6.2.3. Martingale approximations. Another possibility is to use a special martingale approximation of the unknown X . We discretize a separable Hilbert space-valued X with finite-dimensional increasing orthogonal projections P_n on the Cameron-Martin space H of μ_X by setting

$$X_n = \sum_{j=1}^n \langle X, f_j \rangle f_j,$$

where the vectors f_j , $j = 1, \dots, n$ form an orthonormal basis of the finite-dimensional subspace $P_n(H)$ and $\cup_n P_n(H)$ is dense in H . Such martingale approximations were introduced to statistical inverse problems in [94]. Then

$$\mathbf{E}[\langle X_n, \phi \rangle | X_m] = \langle X_m, \phi \rangle$$

when $m \leq n$ and $\phi \in F'$. Hence $\langle X_n, \phi \rangle$ is a martingale. This makes $\|X - X_n\|_F$ a reversed submartingale. It is integrable, since

$$\mathbf{E}[\|X - X_n\|_F^2] \leq \sum_{i=1}^{\infty} \|(I - P_n)e_i\|_F^2 = \|I - P_n\|_{HS}^2,$$

where $\|I - P_n\|_{HS}$ denotes the Hilbert-Schmidt norm of $I - P_n : H \rightarrow F$, and has limit 0 in $L^2(P)$. Hence, $X - X_n$ converge a.s. to zero (see Theorem 10.6.4 in [39]). The almost sure convergence of X_n to X implies the weak convergence of μ_{X_n} to μ .

6.3. Mappings of Gaussian variables. We consider non-linear functions of continuous Gaussian processes as prior models. We start by deforming a Brownian motion $B_t, t \in [0, 1]$, with a continuous function $f : \mathbf{R} \rightarrow \mathbf{R}$ by setting

$$X_t = f(B_t).$$

We check that $X = X_t$ is a random variable having values in a suitable function space F . Obviously, each $X_t, t \in [0, 1]$ is a random variable. The process X_t also inherits sample-continuity from Brownian motion. A natural choice for F is the space $C([0, 1])$ of continuous functions on compact interval $[0, 1]$, equipped with the supremum norm $\|f\|_{\infty} = \sup_{t \in [0, 1]} |f(t)|$. Sample-continuous stochastic processes on $[0, 1]$ are $C([0, 1])$ -valued random variables, since the Borel σ -algebra \mathcal{F} of F coincides with the smallest σ -algebra generated by Dirac's delta functions $\delta_t, t \in [0, 1]$, which are continuous linear forms on $C([0, 1])$ (see Proposition 12.2.2 in [39]).

The process X_t can be discretized by replacing the Brownian motion with its piecewise linear interpolation

$$b_n(t) = \sum_{i=1}^n B_{t_i} \phi_i(t)$$

on $[0, T]$. Here the functions ϕ_i are the usual linear interpolation functions. The mapping $f \mapsto f_n$ is measurable on $C([0, 1])$, and $\lim_{n \rightarrow \infty} f_n = f$ in $C([0, 1])$. Especially, the approximations $b_n(t)$ converge a.s. to B_t in $C([0, T])$ (the sample path of B_t may first be approximated by a C^2 -function). Then $X_n(t) = f(b_n(t))$ converge almost surely to $X_t = f(B_t)$ as $n \rightarrow \infty$ due to the continuity of f . Almost sure convergence implies the weak convergence of the corresponding image measures.

Examples: 1) Take $f(t) = t^2$, i.e. $X_t = B_t^2$ and $X_n(t) = b_n(t)^2$. The positive continuous functions form a measurable set in $C([0, T])$ that has full measure in this case. 2) Take $f(t) = \min(t, 1)$. Then we obtain the approximation $X_n(t) = \min(b_n(t), 1)$ of the bounded function $X_t = \min(B_t, 1)$.

The Brownian motion may be replaced with any other stochastic process whose sample paths are continuous.

6.4. Stochastic integrals. We consider now prior models defined with stochastic integrals

$$X(t) = \int_0^t f(s, \omega) dB_s, \quad t \in [0, T]$$

where B_s is a Brownian motion and $f : [0, T] \times \Omega \rightarrow \mathbf{R}$ is in the class $\mathcal{V}([0, T])$ that satisfies the following conditions. A function $f \in \mathcal{V}([0, T])$ is $\mathcal{B}([0, T] \times \mathcal{F})$ -measurable, $f(t, \omega)$ is \mathcal{F}_t -adapted and $\mathbf{E}[\int_0^t f(t, \omega)^2 dt] < \infty$. Here \mathcal{F}_t is the σ -algebra generated by all B_s , $s \leq t$ (see [111]). Furthermore, we assume that f satisfies

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[\int_0^T \left| \sum_{j=1}^n f(t_{j-1}^{(n)}, \omega) 1_{(t_{j-1}^{(n)}, t_j^{(n)}]}(t) - f(t, \omega) \right|^2 dt \right] = 0,$$

where $0 = t_0^{(n)} < t_1^{(n)} < \dots < t_n^{(n)} = T$ are such that $\max_i (t_i^{(n)} - t_{i-1}^{(n)}) \rightarrow 0$ as $n \rightarrow \infty$.

With probability one, $X(t)$ has continuous sample paths (see [111]). As in the previous section, we may interpret X as $C([0, T])$ -valued random variable. One discrete approximation is to take

$$X_n(t) = \sum_{i=0}^n X_n(t_i^{(n)}) \phi_i(t),$$

where the functions ϕ_i are the linear interpolation functions, and

$$\begin{aligned} X_n(t_i^{(n)}; \omega) &= \sum_{j=1}^i f(\omega, t_{j-1}^{(n)}) (B_{t_j^{(n)}}(\omega) - B_{t_{j-1}^{(n)}}(\omega)) \\ &= \int_0^{t_i^{(n)}} \sum_{j=1}^n f(\omega, t_{j-1}^{(n)}) 1_{(t_{j-1}^{(n)}, t_j^{(n)}]}(t) dB_t(\omega) \end{aligned}$$

are approximations of the stochastic integrals $X(t_i^{(n)})$ for $i \geq 1$ and $X_n(0) = 0$. Then X_n has a subsequence that converges to X on $C([0, T])$. Indeed,

$$\sup_t |X(t) - X_n(t)| \leq \|X(t) - \sum_{i=1}^n X(t_i^{(n)}) \phi_i(t)\|_\infty + \sup_{1 \leq i \leq n} |X(t_i^{(n)}) - X_n(t_i^{(n)})|,$$

where, by Doob's inequality and the Itô isometry the latter term satisfies

$$\begin{aligned} \mathbf{E} \left[\sup_{1 \leq i \leq n} |X_n(t_i^{(n)}) - X(t_i^{(n)})|^2 \right] &\leq C \sup_{1 \leq i \leq n} \mathbf{E} [|X_n(t_i^{(n)}) - X(t_i^{(n)})|^2] \\ &\leq C \mathbf{E} \left[\int_0^T \left| \sum_{j=1}^n f(t_{j-1}^{(n)}, \omega) 1_{(t_{j-1}^{(n)}, t_j^{(n)}]}(t) - f(t, \omega) \right|^2 dt \right], \end{aligned}$$

which converge to zero as $n \rightarrow \infty$. Especially, the sequence $\{\sup_{1 \leq i \leq n} |X_n(t_i^{(n)}) - X(t_i^{(n)})|\}_n$ converges in $L^2(P)$ and has therefore an a.s. convergent subsequence $\{|X_{n_j}(t_i^{(n_j)}) - X(t_i^{(n_j)})|\}_j$. Therefore, $\mu_{X_{n_j}}$ converges weakly to μ_X . Then any subsequence of measures μ_{X_n} has a weakly converging subsequence with the same limit μ_X , and the measures μ_{X_n} converge weakly to μ_X on $C([0, T])$.

6.5. Hyperparametric models. We consider here simple hyperparametric models. Approximations and convergence results in a more complicated case concerning edge-preserving Gaussian hierarchical models were obtained in [63].

Let λ be a Borel measure on \mathbf{R}^d . Let ν_n^t , $n \in \mathbf{N}$ be Borel probability measures on a locally convex Souslin topological vector space F for all $t \in \mathbf{R}^d$ and let $t \mapsto \nu_n^t(U)$ be λ -measurable for all $U \in \mathcal{F}$ and n . If ν_n^t converge weakly to the probability measure ν^t for all t then the hierarchical prior model $\mu_n(U) = \int \nu_n^t(U) d\lambda(t)$ converge weakly. Indeed, if f is a continuous bounded function on F , then

$$\lim_{n \rightarrow \infty} \mu_n(f) = \int \lim_{n \rightarrow \infty} \nu_n^t(f) d\lambda(t) = \mu(f).$$

For example, take $X = \alpha Z$, where Z is zero mean Gaussian with covariance C , and α is an ordinary random variable independent from Z (so-called scale-mixing). We take Z_n to be the linear discretizations $P_n(Z)$ and $X_n = \alpha Z_n$, where $P_n(Z) \rightarrow Z$ a.s. as $n \rightarrow \infty$. We denote the distribution of α with λ . Then the hierarchical prior distributions

$$\mu_{X_n}(U) = \int \mu_{tZ_n}(U) d\lambda(t)$$

converge weakly to μ_X as $n \rightarrow \infty$. This holds especially for sub-Gaussian processes. Moreover, all spherically H -symmetric nonatomic measures are mixtures of Gaussian measure μ_{tZ} , where Z is centered Gaussian with infinite-dimensional Cameron-Martin space H by Theorem 7.4.2 in [12].

If only the distribution of the hyperparameters $\alpha \in \mathbf{R}^d$ is approximated, we may get stronger convergence. Indeed, let Z be any F -valued random variable and let α_n be a sequence of hyperparameters with probability densities $\lambda_n(t)$ on \mathbf{R}^d . We set $X = \alpha Z$ and $X_n = \alpha_n Z$. If the densities $\lambda_n(t)$ converge almost everywhere to the density $\lambda(t)$ of the hyperparameter α and the densities are uniformly bounded, then the hierarchical prior distributions

$$\mu_{X_n}(U) = \int_{\mathbf{R}^d} \mu_{tZ}(U) \lambda_n(t) dt$$

converge in variation to μ_X . Under the conditions of Theorem 4.7, also the corresponding posterior distributions converge in variation.

6.6. Linear discretizations of random variables. In [97], a sequence of random variables X_n having values in a separable Banach space F (equipped with the Borel σ -algebra) is called a *proper linear discretization* of an F -valued random variable X if X_n converge to X weakly in distribution (i.e. $X_n = P_n X$ for a sequence of finite-rank operators P_n on F and $\mu_{\langle X_n, \phi \rangle}$ converge weakly to $\mu_{\langle X, \phi \rangle}$ for all continuous linear functionals $\phi \in F'$). The definition of a proper linear discretization is too weak for the present convergence results. The reason is that the weak convergence in distribution is equivalent to the convergence of the characteristic functionals $\widehat{\mu}_{X_n}(\phi)$ to the characteristic functional $\widehat{\mu}_X(\phi)$ for all $\phi \in F'$ (see Theorem 7.6 in [10]) and the weak convergence of the approximated prior distributions μ_{X_n} is guaranteed if the measures μ_{X_n} are additionally uniformly tight (see Corollary 3.8.5 in [12]). In order to get convergent CM estimates, Lassas et al applied in [96] an enforced condition that $P_n x$ converge in norm to x for all $x \in F$. We follow the Gaussian case [94] and call X_n a *measurable linear discretization* of X if there exists measurable operators P_n having finite-dimensional ranges on F such that $X_n = P_n(X)$ and μ_{X_n} converge weakly to the measure μ_X on F . Moreover, we call X_n a *continuous linear discretization* of X if there exists finite-rank operators (i.e. bounded linear operators with finite-dimensional ranges) P_n on F such that $X_n = P_n X$ and μ_{X_n} converge weakly to the measure μ_X . We discuss shortly the existence of certain linear discretizations.

The notion of a continuous linear discretization is related to the so-called μ -approximation property. Let μ be a Radon probability measure on a separable Banach space F . The space F is said to have the μ -approximation property, if there exists finite-rank operators P_n converging μ -a.s. to identity on F . Moreover, the space F is said to have the *stochastic approximation property* if it has μ -approximation property for every Radon probability measure μ (see [47]). The stochastic μ_X -approximation property gives finite-rank operators P_n , which define continuous linear discretizations of X by $X_n = P_n X$. In [47], it was demonstrated that not all separable Banach spaces have stochastic approximation property. Hence, not all separable Banach space-valued random variables X have almost everywhere converging continuous linear discretizations.

In [47] it was shown that on separable Banach spaces the stochastic approximation property coincides with the existence of a stochastic basis of Herer – a biorthogonal system (e_k, f_k) in (F, F') such that $x = \sum_{k=1}^{\infty} f_k(x) e_k$ for μ -almost every x in F [65]. Candidates of the type $X_n = \sum_{k=1}^n f_k(X) e_k$ are therefore plausible for almost everywhere converging continuous linear discretizations of X . Moreover, if the coefficients $f_k(X)$ are mutually statistically independent and their distributions are equivalent to the Lebesgue measure, then many properties of Gaussian measures hold also for μ_X [126]. For example, measures μ_{X+x_0} are then absolutely continuous with respect to μ_X for every x_0 in the linear span of $\{e_k\}$, the linear span of $\{e_k\}$ has either μ_X -measure zero or one and the topological support of μ_X coincides with the closure of the linear span of $\{e_k\}$ in F .

Continuous linear discretizations of Souslin space-valued unknowns X with property $\mu_X(H) = 1$ for some separable Hilbert space H were considered in [116] (see the discussion in Section 1.3).

In some cases, the prior distribution on a separable Fréchet space F may not be quite what we expect. Okazaki [110] proved a remarkable result that for any

separable Fréchet space F equipped with the Borel σ -algebra, a probability measure μ and a stochastic basis $(e_k, f_k)_{k \in \mathbf{N}}$, there exists a separable Banach space B such that $B \subset F$ continuously, $\mu(B) = 1$ and the stochastic basis $(e_k, f_k)_{k \in \mathbf{N}}$ is actually a Schauder basis for B . The prior distribution μ on F could be replaced with a prior distribution on the Banach space with the Schauder basis. The above result refines Kuelbs' classic result that any Radon measure on a separable Fréchet space has Banach support (see [90], or e.g. Theorem 3.6.5 in [12]).

6.7. Uniformly distributed sequences. We suggest an approximation method that has been historically valued as a competitor to Monte Carlo methods. The best known application of the approximation method is the so-called quasi-Monte Carlo method (see [109]).

Let F be a Hausdorff space and μ a finite Borel measure on F . A sequence $\{x_i\}_{i=1}^\infty \subset F$ is called μ -uniformly distributed if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f(x_i) = \int f(x) d\mu(x)$$

for all continuous and bounded f on F . That is, the average $n^{-1} \sum_{i=1}^n \delta_{x_i}$ of point masses converges weakly to μ . Roughly speaking, it is the law of large numbers with a predetermined sequence.

For any Borel probability measure μ_X on a locally convex Souslin space F , there exists some uniformly distributed sequence $\{x_i\}$ (see Section 8.10 (ix) in [13]). A low dimensional example of such a sequence is the Hammersley sequence for the Lebesgue measure on the unit square [60]. This means that the prior distribution μ_X on (F, \mathcal{F}) may be approximated by measures $\mu_{X_n} = n^{-1} \sum_{i=1}^n \delta_{x_i}$, whose prior information states that a realization of the random variable X_n is one of the values x_i , $i = 1, \dots, n$ and there is no preference between the values x_i , $i = 1, \dots, n$. The uniformly distributed sequence is also a possible tool for interpreting prior information.

7. CONCLUSIONS

The generalized Bayes formula is an efficient tool for obtaining posterior convergence in the weak topology of measures or in the stronger topologies of setwise convergence and convergence in variation (cf. Theorem 4.4 and Theorem 4.7). In the case, when only the hyperparameters of a hierarchical model are approximated, we verified that the posterior distributions converge in variation when the approximations are refined (cf. Section 6.5). In Section 5 we gave examples of applicable non-Gaussian noise models. The explicit expressions of posterior distributions derived in Section 5 for simple non-Gaussian noise models may serve as model cases for further studies on the effects of non-Gaussianity of the noise distribution. In particular, a Kakutani type generalization of the Cameron-Martin formula was derived. We anticipate that this generalization opens a way for a wide class of non-Gaussian noise models in statistical inverse problems, especially when used in connection with the wavelet expansions. Another example demonstrates the surprising fact that the posterior distribution given an infinite-dimensional observation can have significantly simpler expression than the posterior distribution given a corresponding finite-dimensional observation. This suggests that in some cases the infinite-dimensional model could provide new numerical approximations schemes.

It is well-known that the generalized Bayes formula holds when the measures $\mu_{\varepsilon+L(x)}$ are μ_X -almost surely dominated i.e. absolutely continuous with respect to some σ -finite measure for μ_X -almost every $x \in F$. We showed that there is a curious interplay between the continuity of posterior distributions with respect to the observations and the μ_X -a.s. domination of $\mu_{\varepsilon+L(x)}$ (cf. Theorem 2.8 and Remark 4). The continuity of the posterior distribution with respect to observations is only possible in the dominated case, which means that in the undominated cases some posterior distributions have discontinuities. The discontinuities of the posterior distributions may enhance the errors caused by replacing the required sample of $Y_n = L(X_n) + \varepsilon$ in the posterior distribution of X_n given Y_n by the actual observation of $Y = L(X) + \varepsilon$. Moreover, the regularizing effect of the prior distributions on an ill-posed inverse problem could be of limited power.

Continuity of the posterior distributions with respect to the observations has also other roles in statistical inverse problems. It helps to reduce the nonuniqueness of posterior distributions in quite general cases (cf. Theorem 2.7).

In Section 6.6, we discussed the linear discretizations of the unknown X . We remarked that on arbitrary separable Banach spaces there does not always exist a continuous linear discretization by a result of Fonf [47]. The present convergence results, which are written in the same spirit as in [63], are therefore important since they do not require the pointwise convergence of the discretization operators. Beside of continuous linear discretizations, other approximation methods can therefore be used. One of them is a generic method for approximating any prior measure on a locally convex Souslin space with the help of a quasirandom sequence (see Section 6.7).

Finally, we list some directions for generalizing this work. We have not studied the speed of convergence of posterior distributions, which is a very natural question when choosing between different approximation schemes. The generalized Bayes formula gives a good framework for this study. Moreover, we considered only classical noise models i.e. statistically independent noise and unknowns, which enabled us to write a simple expression for the conditional probability of the observation Y given the unknown X . The case of the statistically depended noise and the unknown is not purely theoretical, since often the unknown is approximated with a simple expression and the approximation error is included in the noise term. Convergence of the corresponding posterior distributions is therefore an important topic. Furthermore, we have not discussed what kind of approximating prior distributions could guarantee meaningful convergence of maximum a posteriori (MAP) estimates. Note, that the question is proper for classical noise models that are statistically independent from the unknown since the posterior distribution of X given a sample of $Y = L(X) + \varepsilon$ depends then on the prior model X only through its distribution. The example of Lassas and Siltanen on total variation priors shows that the weak convergence of any approximating prior distributions is, in general, not sufficient as MAP estimates converged then to zero.

There is still a wide class of statistical inverse problems, which are covered neither by the present work nor [94, 116], where the question of posterior convergence remains open.

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